TYPE II NON COMMUTATIVE GEOMETRY. I. DIXMIER TRACE IN VON NEUMANN ALGEBRAS

MOULAY-TAHAR BENAMEUR AND THIERRY FACK

ABSTRACT. We define the notion of Connes-von Neumann spectral triple and consider the associated index problem. We compute the analytic Chern-Connes character of such a generalized spectral triple and prove the corresponding local formula for its Hochschild class. This formula involves the Dixmier trace for II_∞ von Neumann algebras. In the case of foliations, we identify this Dixmier trace with the corresponding measured Wodzicki residue.

CONTENTS

INTRODUCTION

This paper is devoted to an extension, in the framework of type II von Neumann algebras, of the notion of spectral triple introduced by A. Connes [15] in noncommutative geometry. Recall that a spectral triple is a triple $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{H}$ is a Hilbert space, $\mathcal{A}$ is a $*-$subalgebra of $B(\mathcal{H})$ and $D = D^*$ is an unbounded operator on $\mathcal{H}$ whose resolvent is compact and which interracts with $\mathcal{A}$ in a suitable way. A. Connes showed that a large part of Riemannian geometry may be recovered from the study of specific spectral triples. More precisely, let M be a compact oriented (spin) Riemannian $n-$manifold and denote by:

- $\mathcal{H}$ the Hilbert space of L^2-spinors;
- $\mathcal{A}$ the $*-$algebra of smooth functions on M;
- D is the L^2-extension of the Dirac operator on M.

From $(\mathcal{A},\mathcal{H},D)$, we can recover

(1) The Riemannian metric d on M, by the formula:
$$d(x,y)=\mathrm{Sup}\{|f(x)-f(y)|, f\in\mathcal{A} \text{ and } \|[D,f]\|\leq 1\};$$
(2) The smooth structure of $\mathcal{A}$, since we have for any continuous function f on M:
$$f\in\mathcal{A}\Longleftrightarrow f\in\bigcap_{n\geq 1}\mathrm{Dom}(\delta^n),$$
where δ is the unbounded derivation on $B(\mathcal{H})$ defined by $\delta(T):=[|D|,T]$;
(3) The fundamental cycle of M, by the formula:
$$\int_M f^0 df^1\cdots df^n=\mathrm{Tr}_\omega(f^0[D,f^1]\cdots[D,f^n](1+D^2)^{-n/2}),$$
where Tr_ω is the Dixmier trace [23, 15].

In general, spectral triples $(\mathcal{A},\mathcal{H},D)$ give rise to morphisms from the $K-$theory group $K_*(\mathcal{A})$ to the integers. Spectral triples are called even triples when the Hilbert space is $\mathbb{Z}_2-$graded with $\mathcal{A}$ even and D odd for the grading. In this case, the corresponding map on $K-$theory is defined on $K_0(\mathcal{A})$ and assigns to any idempotent $e=e^2\in M_N(\mathcal{A})$, the Fredholm index $\mathrm{Ind}([e(F\otimes 1_N)e]_+)$ of the positive part of $e(F\otimes 1_N)e$, where F is the sign of D. This map $K_0(\mathcal{A})\to\mathbb{Z}$ is given by:
$$\mathrm{Ind}([e(F\otimes 1_N)e]_+)=\phi_{2k}(e,\cdots,e).$$
where ϕ_{2k} (k large enough), is the cyclic $2k-$cocycle on $\mathcal{A}$ defined by
$$\phi_{2k}(a^0,\cdots,a^{2k}):=(-1)^k\,\mathrm{Tr}(\gamma a^0[F,a^1]\cdots[F,a^{2k}]),$$
where γ denotes the grading involution on $\mathcal{H}$. See [17] for more details.

Denote by $\mathrm{Ch}(\mathcal{A},\mathcal{H},D)$ the Chern-Connes character of the spectral triple $(\mathcal{A},\mathcal{H},D)$, i.e. the image of ϕ_{2k} (in the even case) in the periodic cyclic cohomology of $\mathcal{A}$. The computation of $\mathrm{Ch}(\mathcal{A},\mathcal{H},D)$ in terms of local data involving appropriate noncommutative residues is the main step toward the solution of the index problem associated with $(\mathcal{A},\mathcal{H},D)$, and was carried out by A. Connes and H. Moscovici in [19, 20].

In this paper, we consider an extended notion of spectral triples where the operator D is affiliated with some semi-finite von Neumann algebra $\mathcal{M}$. The unitary group of the commutant of $\mathcal{M}$ is thus a symmetry group for D that we want to take into account in order to discuss the associated index problem. The resolvent of D should then be compact with respect to the trace τ of $\mathcal{M}$, whose dimension range can be $[0,+\infty]$. Such triples are called *Connes-von Neumann spectral triples* here. The Murray-von Neumann dimension theory [22] allows to associate a natural index problem to any Connes-von Neumann spectral triple $(\mathcal{A},\mathcal{M},D)$, and our goal is to extend the Connes-Moscovici local index theorem to this framework. This will give a Non Commutative Geometry approach to many well-known index problems involving von Neumann algebras, such as measured foliations [13], Galois coverings [1] or almost periodic operators [12, 40]. Our approach is also motivated by some applications to statistical mechanics that we have in mind [29]. The present paper is a first of a series where we prove the local index theorem for von-Neumann spectral triples.

In order to handle the locality in our discussion of the index problem, we were naturally led to introduce the notion of Dixmier trace for operators affiliated with a semi-finite von Neumann algebra. In particular, we show that the relevent ideal of *infinitesimals* is
$$L^{1,\infty}(\mathcal{M},\tau):=\{T\in\mathcal{M}/\int_0^t\mu_s(T)ds=O(\mathrm{Log}(t)) \text{ when } t\to+\infty\},$$
where $\mu_s(T)$ $(s>0)$ are the generalized $s-$numbers of T [25]. The Dixmier trace of a positive element $T\in L^{1,\infty}(\mathcal{M},\tau)$ is then defined by:
$$\tau_\omega(T):=\lim_{t\to\omega}\left(\frac{1}{\mathrm{Log}(1+t)}\int_0^t\mu_s(T)ds\right),$$

where $\lim_{t\to\omega} f(t)$ is an appropriate conformal invariant limiting process. For a foliation (M, F) with an invariant transverse measure Λ, we recover the transverse integration from the Dirac operator along the leaves D by using our Dixmier trace:

$$\tau_\omega^\Lambda(f|D|^{-p}) = C(p) \int_M f d\Lambda_\nu.$$

In this formula, which may be viewed as the natural extension of Connes' formula [15], Λ_ν is the measure on M associated with Λ [13], and $C(p)$ is a constant depending only on the leaf dimension p.

Let us now describe more precisely the contents of this paper. In the first section, we introduce the Dixmier trace for semi-finite von Neumann algebras and we indicate its relationship with residues of zeta functions. In Section 2, we define the notion of $(p,\infty)-$summable von Neumann spectral triple. We show that such a von Neumann spectral triple $(\mathcal{A}, \mathcal{M}, D)$ gives rise to a *noncommutative integral*

$$\mathcal{M} \ni T \longmapsto \fint T = \tau_\omega(T|D|^{-p}) \in \mathbb{C},$$

which is a hypertrace on the algebra generated by $\mathcal{A}$ and $[D, \mathcal{A}]$. We also express this noncommutative integral by various regularized spectral formulae such as

$$\fint T = \frac{1}{\Gamma(\frac{p}{2}+1)} \lim_{\lambda\to\omega} \left[\frac{1}{\lambda}\tau(Te^{-(D/\lambda^{1/p})^2})\right],$$

which extends the famous Weyl formula. Then, by using the Murray-von Neumann dimension theory, we define the index map associated with $(\mathcal{A}, \mathcal{M}, D)$. This index map is described by an analytical cyclic cocycle on $\mathcal{A}$, the Chern-Connes character of $(\mathcal{A}, \mathcal{M}, D)$, that we identify by proving a generalized Calderon type formula. Section 3 is devoted to a careful analysis of the natural von-Neumann spectral triples associated with order one differential operators along the leaves of a measured smooth foliation on a compact manifold. For an order $-p$ pseudodifferential operator along the leaves $P = (P_L)$, we then compute the Dixmier trace $\tau_\omega(P)$ and show that it coincides with the integrated leafwise Wodzicki residue $\mathrm{res}_L(P_L)$. We also relate the computation of the analytical Chern-Connes character constructed in Section 2, to the solution of the measured index theorem for foliations [13, 34]. In Section 4, we prove a local formula for the image of the Chern-Connes character in Hochschild cohomology. More precisely, we prove that the pairing of this image with Hochschild cycles on $\mathcal{A}$ is the same as the pairing of the Hochschild cocycle ϕ given by the following local formula in the even case:

$$<\phi, \sum_i a_0^i \otimes \cdots \otimes a_p^i> = \sum_i \fint \gamma a_0^i [D, a_1^i] \cdots [D, a_p^i].$$

In the case of measured foliations, the Hochschild class of the Chern-Connes character of the Dirac operator along the leaves coincides with the Ruelle-Sullivan current. When the leaves are four dimensional, our local formula also furnishes, as in [15], a lower bound for the measured Yang-Mills action $\mathrm{YM}^\Lambda(\nabla_E)$ associated with any compatible connection ∇_E on a hermitian vector bundle E over the foliated manifold:

$$| < \frac{c_1(E)^2}{2} - c_2(E), [C_\Lambda] > | \leq \mathrm{YM}^\Lambda(\nabla_E),$$

where C_Λ is the Ruelle-Sullivan current associated with the transverse Λ [38].

For the convenience of the reader, we have also added an appendix on the von Neumann singular numbers.

Preliminary notations. In this paper, we shall denote by $\mathcal{M}$ a von Neumann algebra acting on some Hilbert space. For more informations about von Neumann algebras, we refer to [22]. All the von Neumann algebras considered in this paper will be of type II_∞, i.e. they are semi-finite and properly infinite [22]. For such a von Neumann algebra $\mathcal{M}$, let τ be a normal faithful semi-finite trace. We denote for $p \geq 1$, by $L^p(\mathcal{M}, \tau)$ the completion of the space of all $T \in \mathcal{M}$ such that $\tau(|T|^p) < +\infty$ for the norm

$$\|T\|_p := \tau(|T|^p)^{1/p}.$$

See Appendix A for more details. From Section 2 on, $\mathcal{A}$ denotes a $*-$subalgebra of $\mathcal{M}$. The unitary group of the commutant $\mathcal{M}'$ of $\mathcal{M}$, is thus a symmetry group for $\mathcal{A}$.

Acknowledgements. We would like to thank A. Connes for his comments and encouragements. We also wish to thank J. Bellissard, A. Carey, J. Heitsch, V. Nistor, J. Phillips, R. Prinzis, G. Skandalis, F. Sukochev, J. Varilly and S. Vassout for several helpful discussions.

1. Dixmier traces on von Neumann algebras

1.1. **Review of the classical case.** To show the existence of a non normal trace on the algebra $B(H)$ of all bounded operators on an infinite dimensional separable Hilbert space H, J. Dixmier [23] constructed in 1966 a unitarily invariant state Tr_ω on the ideal $L^{1,\infty}(H)$ of all compact operators T on H whose singular numbers $\mu_0(T) \geq \mu_1(T) \geq ...$ satisfy

$$\sum_{k=0}^{n-1} \mu_k(T) = O(\mathrm{Log}(n)) \text{ when } n \to +\infty.$$

For a positive operator $T \in L^{1,\infty}(H)$, the trace $\mathrm{Tr}_\omega(T)$ was defined as a renormalized limit:

$$\mathrm{Tr}_\omega(T) = \lim_\omega \frac{1}{\mathrm{Log}(N)} \sum_{n=0}^{N-1} \mu_n(T),$$

where $a = (a_n)_{n\geq 0} \to \omega(a) = \lim_\omega(a_n)$ denotes a state on $l^\infty(\mathbb{N})$ which vanishes on $c_0(\mathbb{N})$ and satisfies $\lim_\omega a_n = \lim_\omega a_{2n}$. For $T \in L^{1,\infty}(H)$, it was proved later (cf [24]) that $\mathrm{Tr}_\omega(T)$ only depends on the spectral measure of T on $Sp(T) \setminus \{0\}$.

Let T be a scalar pseudodifferential operator of order $-n$ on a closed Riemannian n-manifold M. A. Connes noticed in [14] that T is in the Dixmier ideal $L^{1,\infty}(H)$, where $H = L^2(M)$, and he proved that $\mathrm{Tr}_\omega(T)$ coincides up to a constant with the Wodzicki residue. More precisely, we have for such an operator T with principal symbol $\sigma_{-n}(T)$ and Schwartz kernel $k(x,y)$:

$$\mathrm{Tr}_\omega(T) = \frac{1}{n(2\pi)^n} \int_{\mathbb{S}^*M} \sigma_{-n}(T)(x,\xi) dx d\xi = \int a(x) = (1/n) \times Res_{z=0} \mathrm{Tr}(T\Delta^{-z}),$$

where Δ is any positive invertible differential operator of order 1 on M, $dxd\xi$ is the natural Liouville measure on the cosphere bundle $\mathbb{S}^*M$, and the density a satisfies:

$$k(x,y) = a(x)\,\mathrm{Log}(|x-y|) + O(1)$$

near the diagonal. These formulae trivially extend to the case of pseudodifferential operators with coefficients in a vector bundle. In particular, taking Δ with principal symbol $\sigma(\Delta)(x,\xi) = |\xi|$ for a given Riemannian metric on M, we get:

$$\int_M f dvol = \frac{n \times (2\pi)^n}{vol(\mathbb{S}^{n-1})} \times \mathrm{Tr}_\omega(f\Delta^{-n}), \quad f \in C^\infty(M).$$

This led A. Connes to introduce the Dixmier trace Tr_ω as the correct operator theoretical substitute for integration of infinitesimals of order one in Non Commutative Geometry. We shall now extend Connes' definition of the Dixmier trace to the case of semi-finite traces on infinite von Neumann algebras.

1.2. **Dixmier ideal in a von Neumann algebra.** Let us denote by $\mathcal{M}$ an infinite semi-finite von Neumann algebra acting on a Hilbert space H and equipped with a faithful normal semi-finite trace τ. For any τ-measurable operator T in H, denote by

$$\mu_t(T) = \mathrm{Inf}_{t>0}\{\|TE\|, E = E^* = E^2 \in \mathcal{M}, \tau(1-E) \leq t\}$$

the tth generalized s-number of T (See Appendix A.1 for more details).

An element $T \in \mathcal{M}$ is called τ-compact if $\lim_{t\to\infty} \mu_t(T) = 0$. The set of all τ-compact elements in $\mathcal{M}$ is a norm closed ideal of $\mathcal{M}$ that we shall denote by $\mathcal{K}(\mathcal{M},\tau)$. By [25][page 304], the ideal $\mathcal{K}(\mathcal{M},\tau)$ is the norm closure of the ideal $\mathcal{R}(\mathcal{M},\tau)$ of all elements X in $\mathcal{M}$ whose final support $r(X) = \mathrm{Supp}(X^*)$ satisfies $\tau(r(X)) < \infty$.

Definition 1. An element $T \in \mathcal{M}$ is called of Dixmier trace class (with respect to τ) if:

$$\int_0^t \mu_s(T)ds = O(\mathrm{Log}(1+t)), \text{ when } t \to +\infty.$$

In the sequel, we shall set $\sigma_t(T) := \int_0^t \mu_s(T)ds$. The set of Dixmier trace class operators is a vector space that we shall denote by $L^{1,\infty}(\mathcal{M}, \tau)$. It is a Banach space for the norm

$$\|T\|_{1,\infty} = \mathrm{Sup}_{t>0} \frac{\sigma_t(T)}{\mathrm{Log}(t+1)},$$

and an ideal in $\mathcal{M}$ which contains $L^1(\mathcal{M}, \tau) \cap \mathcal{M}$. Note that we have for any $T \in L^{1,\infty}(\mathcal{M}, \tau)$ and $t > 0$:

$$\mu_t(T) \leq \frac{\sigma_t(T)}{t} \leq \|T\|_{1,\infty} \frac{\mathrm{Log}(1+t)}{t},$$

so that we get for any $\epsilon > 0$:

$$L^1(\mathcal{M}, \tau) \cap \mathcal{M} \subset L^{1,\infty}(\mathcal{M}, \tau) \subset L^{1+\epsilon}(\mathcal{M}, \tau) \cap \mathcal{M} \subset \mathcal{K}(\mathcal{M}, \tau).$$

We shall set for $1 < p < +\infty$:

$$L^{p,\infty}(\mathcal{M}, \tau) := \{T \in \mathcal{M} / \sigma_t(T) = O(t^{1-\frac{1}{p}})\}.$$

For $T \in L^{p,\infty}(\mathcal{M}, \tau)$ with $1 < p < +\infty$, we have $\mu_t(T) = O(t^{-1/p})$, it follows that $|T|^p \in L^{1,\infty}(\mathcal{M}, \tau)$.

1.3. **Dixmier trace.** The general notion of singular traces on von Neumann algebras has been introduced in [27] and used in [28] to investigate the Novikov-Shubin invariants. For our purpose, we shall consider here Dixmier traces defined in terms of singular numbers.

To this end, we shall use limiting processes

$$\omega : L^\infty([0, +\infty[) \ni f \longmapsto \omega(f) \in \mathbb{C}.$$

More precisely, ω is a linear form on $L^\infty([0, +\infty[))$ satisfying the following conditions:

(1) $\lim \text{ ess inf }_{t\to+\infty} f(t) \leq \omega(f) \leq \lim \text{ ess sup }_{t\to+\infty} f(t)$;
(2) $\omega(f) = \omega(M(f))$, where $M(f)(t) = \frac{1}{\mathrm{Log}(t)} \int_1^t f(s)ds/s$.

Note that $M(f)$ is continuous and bounded on $[1, +\infty[$ for any bounded function f. The first condition implies that ω is a state on $L^\infty([0, +\infty[))$ that vanishes on $C_0([0, +\infty[)$, and the second condition implies the following scale-invariance property:

(3) For any $\lambda > 0$ and any $f \in L^\infty([0, +\infty[)$, we have $\omega(f) = \omega(f_\lambda)$, where $f_\lambda(t) = f(\lambda t)$.

The existence of a limiting process satisfying the two conditions is obvious. Indeed, let ϕ be a state on the C^*-algebra $C_b(\mathbb{R}_+)$ vanishing on $C_0(\mathbb{R}_+)$ and set for $a = (a_n)_{n\geq 0} \in l^\infty(\mathbb{N})$:

$$\lim_\phi(a) = \phi(\tilde{a}),$$

where $\tilde{a} \in C_b([0, +\infty[)$ is the piecewise linear function defined by $\tilde{a}(n) = \frac{a_0+\cdots+a_n}{n+1}$ for any $n \geq 0$. Then,

$$\omega(f) := \lim_\phi(\phi(M^n(f)), \tag{1}$$

is a limiting process satisfying (1) and (2). As in [15], we shall write $\omega(f) = \lim_{t\to\omega} f(t)$. Note that we have, for such a limiting process:

$$(4) \quad |\lim_{t\to\omega} f(t) - \alpha \lim_{t\to\omega} f(\lambda t^\alpha)| \leq (1-\alpha)\|f\|_\infty,$$

for any $\alpha \in]0, 1[$, any $\lambda > 0$ and any $f \in L^\infty([0, +\infty))$. This follows from the estimate:

$$|\frac{1}{\mathrm{Log}(t)} \int_1^t f(s)ds/s - \frac{\alpha}{\mathrm{Log}(t)} \int_1^t f(\lambda s^\alpha)ds/s| \leq \|f\|_\infty (1 - \alpha + 2) \frac{|\mathrm{Log}(\lambda)|}{\mathrm{Log}(t)}.$$

Definition 2. For any limiting process ω as above, the Dixmier trace $\tau_\omega(T)$ of a positive operator $T \in L^{1,\infty}(M,\tau)$ is defined by:

$$\tau_\omega(T) := \omega\left(t \to \frac{\sigma_t(T)}{\mathrm{Log}(t+1)}\right).$$

Let us point out that this definition depends on the choice of ω. However, if $l = \lim_{t\to+\infty}[\frac{1}{\mathrm{Log}(t)}\int_a^t \frac{\sigma_s(T)}{log(s)}\frac{ds}{s}]$ exists, then $\tau_\omega(T) = l$.

From now on we fix a limiting process $\omega(f) = \lim_{t\to\omega} f(t)$. As in the classical case (cf [19]) one may prove that we have for any positive operators $T, S \in L^{1,\infty}(\mathcal{M},\tau)$:

$$\tau_\omega(T+S) = \tau_\omega(T) + \tau_\omega(S).$$

This enables to extend τ_ω to a positive linear form on the Dixmier ideal $L^{1,\infty}(\mathcal{M},\tau)$. A classical argument (cf [15]) shows that $\tau_\omega(ST) = \tau_\omega(TS)$ for any $T \in L^{1,\infty}(\mathcal{M},\tau)$ and any $S \in \mathcal{M}$.

Note that $\tau_\omega(T) = 0$ when $T \in \mathcal{M} \cap L^1(\mathcal{M},\tau)$. In the same way, it is easy to check for instance by using the inequality

$$\int_0^t \mu_s(TS)ds \leq \int_0^t \mu_s(T)\mu_s(S)ds,$$

(See [25]), that $\tau_\omega(TS) = 0$ for any $T, S \in L^{1,\infty}(\mathcal{M},\tau)$.

Theorem 1. *Let $A \in L^{1,\infty}(\mathcal{M},\tau)$ be a positive element and set $E_t := 1_{]t,+\infty)}(A)$ for any $t > 0$. Then for any $T \in \mathcal{M}$, the functions*

$$t \mapsto \frac{1}{\mathrm{Log}(t+1)}\tau(TE_{\mu_t(A)}A) \text{ and } t \mapsto \frac{1}{\mathrm{Log}(t+1)}\tau(TE_{1/t}A),$$

are bounded and we have:

$$\begin{aligned}\tau_\omega(TA) &= \lim_{t\to\omega}\frac{1}{\mathrm{Log}(t+1)}\tau(TE_{\mu_t(A)}A)\\ &= \lim_{t\to\omega}\frac{1}{\mathrm{Log}(t+1)}\tau(TE_{1/t}A)\end{aligned}$$

Proof. Since $\mu_s(A) \to 0$ when $s \to +\infty$, we have:

$$|\tau(TE_{\mu_t(A)}A)| \leq \|T\| \int_{\{s>0/\mu_s(A)>\mu_t(A)\}} \mu_s(A)ds \leq \|T\| \int_0^t \mu_s(A)ds.$$

It follows that $t \mapsto \frac{1}{\mathrm{Log}(t+1)}\tau(TE_{\mu_t(A)}A)$ is bounded. To prove that $\frac{1}{\mathrm{Log}(t+1)}\tau(TE_{1/t}A)$ is bounded, we may assume w.l.o.g. that $A \neq 0$. For any $t > \frac{1}{\|A\|}$, let $s(t)$ be the unique $s \geq 0$ such that:

$$\mu_s(A) \leq 1/t \text{ and } \mu_{s-\epsilon}(A) > 1/t, \forall \epsilon > 0.$$

Since $A \in L^{1,\infty}(\mathcal{M},\tau)$, there exists a constant $C > 0$ such that $\mu_s(A) \leq C\frac{\mathrm{Log}(s+1)}{s+1}$ for any $s > 0$. Set $u(t) = s(t) + 1$. From the inequality

$$1/t < \mu_{s(t)-\epsilon}(A) \leq C\frac{\mathrm{Log}(s(t)-\epsilon+1)}{s(t)-\epsilon+1},$$

we deduce by letting $\epsilon \to 0$:

$$u(t) \leq Ct\,\mathrm{Log}(u(t)) \quad \text{ for } t > 1/\|A\|.$$

We claim that this implies the existence of a constant $K > 0$ such that:

$$u(t) \leq Kt\,\mathrm{Log}(t+1) \quad \text{ for } t > 1/\|A\|. \tag{2}$$

Since Equation (2) is obvious when u is bounded, we may assume again w.l.o.g. that it is not the case, and hence $\lim_{t\to+\infty} u(t) = +\infty$ since u is non-decreasing. Assume that (2) is false. Then there exists for any integer $n > 0$ a real number $t_n > 1/\|A\|$ such that:

$$u(t_n) > nt_n\,\mathrm{Log}(t_n+1) > \frac{n}{\|A\|}\mathrm{Log}(\frac{1}{\|A\|}+1).$$

It follows that $u(t_n) \to +\infty$ when $n \to +\infty$ and hence $t_n \to +\infty$. Write $u(t_n) = \epsilon_n t_n \operatorname{Log}(t_n + 1)$ where $\epsilon_n > n$. We have:

$$C \geq \frac{u(t_n)}{t_n \operatorname{Log}(u(t_n))} = \frac{\epsilon_n \operatorname{Log}(t_n+1)}{\operatorname{Log}(t_n) + \operatorname{Log}(\operatorname{Log}(t_n+1)) + \operatorname{Log}(\epsilon_n)} \longrightarrow +\infty,$$

a contradiction. So, equation (2) is true and we have for $t > 1/\|A\|$:

$$\begin{aligned} |\tau(TE_{1/t}A)| &\leq \|T\|\tau(E_{1/t}A) \\ &= \|T\| \int_0^{s(t)} \mu_s(A)ds \\ &\leq \|T\| \int_0^{Kt\operatorname{Log}(t+1)} \mu_s(A)ds \\ &= O(\operatorname{Log}(t \operatorname{Log}(t+1)), \end{aligned}$$

a fact which implies that $t \mapsto \frac{1}{\operatorname{Log}(t+1)}\tau(TE_{1/t}A)$ is bounded.

Let us set, for any $T \in \mathcal{M}$:

$$\varphi(T) := \tau_\omega(TA), \psi(T) := \lim_{t\to\omega} \frac{\tau(TE_{\mu_t(A)}A)}{\operatorname{Log}(1+t)} \text{ and } \theta(T) := \lim_{t\to\omega} \frac{\tau(TE_{1/t}A)}{\operatorname{Log}(1+t)}.$$

We thus define positive linear forms on $\mathcal{M}$ and we claim that

$$\theta \leq \psi \leq \varphi. \tag{3}$$

For simplicity, set $P_t = E_{\mu_t(A)}$ and $Q_t = E_{1/t}$ for $t > 0$. Since we have $\mu_t(A) \leq C\frac{\operatorname{Log}(1+t)}{1+t}$, there exists for any $\alpha \in]0,1[$ a constant $C_\alpha > 0$ such that $\mu_t(A) \leq t^{\alpha-1}/C_\alpha$ for any $t > 0$. Hence:

$$Q_{C_\alpha t^{1-\alpha}} A \leq P_t A.$$

For any $T \in \mathcal{M}$ with $T \geq 0$, we deduce that:

$$\tau(TQ_{C_\alpha t^{1-\alpha}}A) \leq \tau(TP_tA),$$

and hence, since $\frac{\operatorname{Log}(t+1)}{\operatorname{Log}(C_\alpha t^{1-\alpha}+1)} \to \frac{1}{1-\alpha}$ when $t \to +\infty$:

$$(1-\alpha)\lim_{t\to\omega} f(C_\alpha t^{1-\alpha}) \leq \lim_{t\to\omega} \frac{\tau(TP_tA)}{\operatorname{Log}(t+1)} = \psi(T),$$

where $f(t) = \frac{\tau(TQ_tA)}{\operatorname{Log}(t+1)}$. But we have:

$$|\lim_{t\to\omega} f(t) - (1-\alpha)\lim_{t\to\omega} f(C_\alpha t^{1-\alpha})| \leq \alpha\|f\|_\infty,$$

so that we get by letting $\alpha \to 0$:

$$\theta(T) = \lim_{t\to\omega} f(t) \leq \psi(T).$$

On the other hand, since we have $\tau(P_t) \leq t$ (cf [26][Prop. 2.2, p. 274]), we get:

$$\tau(TP_tA) = \tau(A^{1/2}TA^{1/2}P_t) = \int_0^{+\infty} \mu_s(A^{1/2}TA^{1/2}P_t)ds = \int_0^t \mu_s(A^{1/2}TA^{1/2}P_t)ds.$$

The last equality is deduced from the fact that [26][Lemma 2.6, p. 277]:

$$\mu_s(A^{1/2}TA^{1/2}P_t) = 0, \quad \forall s \geq \tau(P_t),$$

We thus have:

$$\tau(TP_tA) \leq \|P_t\| \int_0^t \mu_s(A^{1/2}TA^{1/2})ds,$$

and hence:

$$\psi(T) \leq \tau_\omega(A^{1/2}TA^{1/2}) = \tau_\omega(TA) = \varphi(T).$$

This proves (3). To show that $\theta = \psi = \varphi$ and achieve the proof of the theorem, it suffices to prove that $\varphi(I) \leq \theta(I)$. But we have by [26][p. 289]:

$$\sigma_t(A) = \mathrm{Inf}\{\|T_1\|_1 + t\|T_2\|/A = T_1 + T_2\},$$

so that we get by taking $T_1 = Q_tA$ and $T_2 = A - T_1$:

$$\sigma_t(A) \leq \tau(Q_tA) + t\|1_{[0,1/t]}(A)\| \leq \tau(Q_tA) + 1.$$

It follows that:

$$\varphi(I) = \lim_{t\to\omega} \frac{\sigma_t(A)}{\mathrm{Log}(t+1)} \leq \lim_{t\to\omega} \frac{\tau(Q_tA)}{\mathrm{Log}(t+1)} = \theta(I).$$

□

Remark 1. In the above proof, the property $\omega(f) = \omega(M(f))$ was used to prove that $\theta \leq \psi$. It is worthpointing out that in most of the examples we have in mind, the operator A will actually satisfy the following better estimate:

$$\mu_t(A) = O(1/t).$$

In this case, the above theorem remains true for limiting processes $\omega(f) = \lim_\omega f(t)$ only satisfying the weaker scale invariance property $\omega(f_\lambda) = \omega(f)$. Indeed, if $t > 0$ and if A satisfies the relation $\mu_t(A) = C/t$, we have

$$Q_{t/C}A \leq P_tA,$$

and hence

$$\tau(TQ_{t/C}A) \leq \tau(TP_tA).$$

So if we only assume that $\lim_{t\to\omega} f(t) = \lim_{t\to\omega} f(t/C)$, we get

$$\theta(T) = \lim_{t\to\omega} \frac{\tau(TQ_tA)}{\mathrm{Log}(t)} = \lim_{t\to\omega} \frac{\tau(TQ_{t/C}A)}{\mathrm{Log}(t/C)} \leq \lim_{t\to\omega} \tau(TP_tA) = \psi(T).$$

1.4. **Dixmier trace and residue of zeta functions.** Let us first define the zeta function of a positive self-adjoint τ-discrete operator T in an infinite semi-finite von Neumann algebra $\mathcal{A}$ acting on H and equipped with a normal faithful positive trace τ.

Definition 3. A positive self-adjoint τ-measurable operator T on H is called τ-discrete if $(T-\lambda)^{-1} \in \mathcal{K}(\mathcal{M}, \tau)$ for any $\lambda < 0$.

It may be proved (cf [37][page 48]) that $T = \int_0^{+\infty} \lambda dE_\lambda$ is τ-discrete if and only if one of the two following properties holds:
(i) $\forall \lambda \in \mathbb{R}$, $\tau(E_\lambda) < +\infty$;
(ii) $\exists \lambda_0 < 0$ such that $(T - \lambda_0)^{-1} \in \mathcal{K}(\mathcal{M}, \tau)$.

For such a positive τ-discrete operator T, the function

$$N_T(\lambda) := \tau(E_\lambda)$$

is well defined on $\mathbb{R}$. Moreover, it is nondecreasing, positive and right continuous.

Definition 4. Let $T = \int_\epsilon^{+\infty} \lambda dE_\lambda$ be a positive self-adjoint τ-discrete operator with spectrum in $[\epsilon, +\infty)$, where $\epsilon > 0$. The zeta function ζ_T of T is defined by:

$$\zeta_T(z) := \int_\epsilon^{+\infty} \lambda^z dN_T(\lambda),$$

for any complex argument z such that the above integral converges.

Since we have for any $R > 1$:

$$\int_1^R \lambda^z dN_T(\lambda) = \int_0^{\mathrm{Log}(R)} e^{zt} d\alpha(t),$$

where $\alpha(t) = N_T(e^t)$, we know from the classical Laplace-Stieltjes transform theory (see [43]) that the integral

$$\int_\epsilon^{+\infty} \lambda^z dN_T(\lambda)$$

converges for $Re(z) < -d_T$ and diverges for $Re(z) > -d_T$, where:

$$d_T := \overline{\lim}_{t\to+\infty} \frac{\mathrm{Log}(\alpha(t))}{t} = \overline{\lim}_{\lambda\to+\infty} \frac{\mathrm{Log}(N_T(\lambda))}{\mathrm{Log}(\lambda)}.$$

Moreover, ζ_T is analytic in the half-plane $\{Re(z) < -d_T\}$ and $z = -d_T$ is a singularity of ζ_T if $d_T < +\infty$.

We also have:

$$d_T = \mathrm{Inf}\{\mu \in \mathbb{R}, T^z \in L^1(\mathcal{M}, \tau) \text{ for } Re(z) < -\mu\}.$$

Indeed, it follows from the normality of the trace that:

$$\mathrm{Sup}_{R>0} \int_0^R \lambda^x dN_T(\lambda) = \mathrm{Sup}_{R>0}\, \tau(E_R(T)T^x) = \tau(T^x), \qquad \text{for any } x \in \mathbb{R}.$$

In particular:

$$d_T < +\infty \Longleftrightarrow \exists x \leq 0 \text{ such that } T^x \in L^1(\mathcal{M}, \tau).$$

Sometimes, d_T is called the quantum τ-dimension of the operator T.

Theorem 2. *Let T be a positive τ-discrete operator with spectrum in $[\epsilon, +\infty]$, $\epsilon > 0$. If $0 < d_T < +\infty$, the following conditions are equivalent:*

(i) $(x + d_T)\zeta_T(x) \to A$ when $x \to -d_T, x \in (-\infty, -d_T[$;

(ii) $T^{-d_T} \in L^{1,\infty}(\mathcal{M}, \tau)$ and

$$\tau_\omega(T^{-d_T}) = \lim_{t\to+\infty} \frac{1}{\mathrm{Log}(1+t)} \int_0^t \mu_s(T^{-d_T}) ds = -\frac{A}{d_T}. \tag{4}$$

Proof. This theorem easily follows from the equivalence, for any positive and non increasing function f on $[0, +\infty[$ such that $\int_0^\infty f(t)^s dt < +\infty$ for any $s > 1$, of the two following assertions:

(a) $(s-1)\int_0^{+\infty} f(t)^s dt \to L$ when $s \to 1^+$;

(b) $\frac{1}{\mathrm{Log}(u)} \int_0^u f(s) ds \to L$ when $u \to +\infty$.

The proof of this equivalence uses classical abelian and tauberian theorems. For completeness, we give a proof based on Proposition 1 (see also [37]).

(i) $\Rightarrow$ (ii): Assume that $Re(z) < -d_T$, and make the change of variable $\lambda = e^{u/d_T}$ in the integral defining $\zeta_T(z)$. We get $\zeta_T(z) = f_1(-z/d_T) + f_2(z)$ where

$$f_1(z) = \int_0^{+\infty} e^{-zu} d\phi(u), \text{ with } \phi(u) = N_T(e^{u/d_T}) 1_{(d_T,+\infty)}(u) \text{ and } f_2(z) = \int_\epsilon^e \lambda^z dN_T(\lambda).$$

The function f_2 is entire, while f_1 only converges for $Re(z) > 1$ and satisfies

$$\lim_{x\to 1^+} (x-1) f_1(x) = -A/d_T.$$

Setting $g(z) = f_1(z+1)$ we then obtain

$$g(z) = \int_0^\infty e^{-zt} d\beta(t) \text{ where } \beta(t) = \int_0^t e^{-u} d\phi(u) \text{ for any } t \geq 0.$$

Then $g(z)$ is a convergent integral for $Re(z) > 0$ such that $\lim_{x\to 0^+} xg(x) = -A/d_T$. By the Hardy-Littlewood tauberian theorem, we get:

$$-\frac{A}{d_T} = \lim_{t\to+\infty} \frac{\beta(t)}{t} = \lim_{t\to+\infty} \frac{1}{t}\int_e^{e^{t/d_T}} \lambda^{-d_T} dN_T(\lambda),$$

and hence:

$$-\frac{A}{d_T} = \lim_{t\to+\infty} \frac{1}{\mathrm{Log}(t)} \int_{(1/t,+\infty)} \lambda dN_T(\lambda^{-1/d_T}) = \lim_{t\to+\infty} \frac{1}{\mathrm{Log}(t)} \tau(1_{(1/t,\infty)}(T^{-d_T})T^{-d_T}).$$

The result follows from Proposition 1.

(i) $\Leftarrow$ (ii): Let us first assume that $A > 0$. Fix $\epsilon > 0$ with $\epsilon < \frac{A}{d_T}$ and choose $M > 0$ such that:

$$|\frac{1}{\mathrm{Log}(1+t)} \int_0^t \mu_s(T^{-d_T})ds + A/d_T| \leq \epsilon, \quad \forall t \geq M.$$

We have for any $t \geq M$,

$$(-\frac{A}{d_T} - \epsilon)\int_0^t \frac{ds}{1+s} \leq \int_0^t \mu_s(T^{-d_T})ds \leq (-\frac{A}{d_T} + \epsilon)\int_0^t \frac{ds}{1+s}. \tag{5}$$

If these inequalities were true for any $t > 0$ we would have:

$$\forall t > 0, \int_0^t f(s)ds \leq \int_0^t h(s)ds \leq \int_0^t g(s)ds,$$

where

$$f(t) := (\frac{-A}{d_T} - \epsilon)\frac{1}{1+t}, g(t) := (\frac{-A}{d_T} + \epsilon)\frac{1}{1+t} \text{ and } h(t) := \mu_t(T^{-d_T})$$

are non increasing positive functions. Using Polya's inequality we would get:

$$\forall t > 0, \forall a \geq 1, \int_0^t (f(s))^a ds \leq \int_0^t (h(s))^a ds \leq \int_0^t (g(s))^a ds,$$

and hence by letting $t \to \infty$:

$$\frac{(-A/d_T - \epsilon)^a}{a-1} \leq \int_0^{+\infty} \mu_s(T^{-d_T a})ds \leq \frac{(-A/d_T + \epsilon)^a}{a-1} \quad \text{for any } a > 1.$$

Setting $a = -x/d_T$ with $x < -d_T$ and multiplying the above inequalities by $-x - d_T > 0$, we get the result when $x \to -d_T$.

Now since the inequaliy (5) is only true for $t \geq M$, we take f_1, g_1 and h_1 equal to f, g and h for $t \geq M$, and for $t < M$:

$$f_1(t) := \frac{1}{M}\int_0^M f(v)dv, g_1(t) := \frac{1}{M}\int_0^M g(v)dv \text{ and } h_1(t) := \frac{1}{M}\int_0^M h(v)dv.$$

Since f_1, g_1 and h_1 are nonincreasing positive functions such that:

$$\forall t > 0, \int_0^t f_1(s)ds \leq \int_0^t h_1(s)ds \leq \int_0^t g_1(s)ds,$$

we now use Polya's inequality to get the conclusion. Indeed, the additional constants that arise with these modifications do not change the final computation of the residue of ζ_T at $-d_T$.
If $A = 0$ the same proof still works, replacing f by the zero function. □

To end this paragraph, we point out the strong relation between the Dixmier trace and the asymptotics of the spectrum. For instance we have:

Proposition 1. [37] *Let T be as in Theorem 2. Assume that there exists $\delta > 0$ such that ζ_T admits a meromorphic extension to $\{Re(z) < -d_T + \delta\}$ with a simple pole at $z = -d_T$. Then we have:*

$$\lim_{\lambda\to+\infty} \frac{N_T(\lambda)}{\lambda^{d_T}} = -\frac{Res_{-d_T}(\zeta_T)}{d_T} = \tau_\omega(T^{-d_T}).$$

This proposition is a simple consequence of the Ikehara tauberian theorem.

2. The von-Neumann index problem

The data proposed by A. Connes to define a "geometry" is a triple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is a $*$-algebra represented in a Hilbert space $\mathcal{H}$ and D is an unbounded densely defined self-adjoint operator with a summability condition. To work with such a spectral triple, A. Connes introduces some constraints on the interaction between D and $\mathcal{A}$. This formalism has been very fruitful especially in exploring index theory for singular spaces. We extend in this section some known results in non commutative geometry to the setting of von-Neumann algebras, whixh will allow us to reach the index theory of measured families of geometries.

2.1. **Von Neumann spectral triples.** In view of polynomial formulae, we shall restrict ourselves to finite dimensional spectral triples. The general case can be treated similarly extending the notion of θ-summability [6].

Definition 5. By a $p-$summable von Neumann spectral triple we shall mean a triple $(\mathcal{A}, \mathcal{M}, D)$ where $\mathcal{M} \subset B(H)$ is a von Neumann algebra faithfully represented in a Hilbert space H and endowed with a (positive) normal semi-finite faithful trace τ, $\mathcal{A}$ is a $*$-subalgebra of the von Neumann algebra $\mathcal{M}$, and D is a τ-measurable self-adjoint operator such that:

(i) $\forall a \in \mathcal{A}$, the operator $a(D+i)^{-1}$ belongs to the Dixmier ideal $L^{p,\infty}(\mathcal{M}, \tau)$;

(ii) Every element $a \in \mathcal{A}$ preserves the domain of D and the commutator $[D, a]$ belongs to $\mathcal{M}$;

(iii) For any $a \in \mathcal{A}$, the operators a and $[D, a]$ belong to $\cap_{n\in\mathbb{N}} Dom(\delta^n)$, where δ is the unbounded derivation of $\mathcal{M}$ given by $\delta(b) = [|D|, b]$.

When $\mathcal{M}$ is $\mathbb{Z}_2$-graded with $\mathcal{A}$ even and D odd, we say that the von Neumann-spectral triple is even and denote by $\gamma \in \mathcal{M}$ the grading involution. Otherwise, the triple is called an odd triple.

Examples. (1) Let M be a compact Riemannian manifold of dimension n. Let D be a generalized Dirac operator. Then we set $\mathcal{A} = C^\infty(M)$, $\mathcal{M} = B(H), D)$ where H is the L^2-space of corresponding generalized spinors. With the operator D we get an n-summable von Neumann spectral triple. It is even when n is even. As proved by A. Connes, one completely recovers the Gauss-Riemann calculus on M from the study of the spectral triple $(\mathcal{A}, \mathcal{M}, D)$ [15].

(2) Let (M, F) be a compact foliated manifold with a holonomy invariant transverse measure Λ and let $p \geq 1$ be the dimension of the leaves of (V, F). Let D be a generalized Dirac operator along the leaves of (M, F) acting on sections of a hermitian vector bundle E. Denote by $W^*_\Lambda(M, F; E)$ the von Neumann algebra associated with Λ and E (see Section 3). The holonomy invariant transverse measure Λ gives rise to a semi-finite normal trace τ_Λ on $W^*_\Lambda(M, F; E)$ by the formula:

$$\tau_\Lambda(T) := \int_{M/F} \mathrm{Trace}(T_L) d\Lambda(L).$$

The triple $(\mathcal{A} = C^\infty(M), W^*_\Lambda(M, F; E), D)$ is then a $p-$summable von Neumann spectral triple which is not a type I spectral triple (See Section 3 for more details). Again the triple is even when p is even.

(3) Let $\Gamma \hookrightarrow \tilde{M} \to M$ be a Galois cover of a compact $n-$dimensional manifold M. Let D be the Γ cover of a generalized Dirac operator on M. Consider the von Neumann algebra $\mathcal{M}$ of bounded Γ-invariant operators defined by Atiyah in [1], with its natural trace Tr_Γ. Then $(\mathcal{A} = C^\infty(M), \mathcal{M}, D)$ is an n-summable von Neumann spectral triple (See [6] for more details), which is even when n is even.

(4) Let $D = \sum_i a_i(x) D^i_x + b(x)$ be a first order uniformly elliptic differential operator with almost periodic coefficients on $\mathbb{R}^n$. The index of such an operator can be defined by considering a spectral triple $(\mathcal{A}, \mathcal{M}, \tilde{D})$ that we shall briefly describe, see [39]. The operator $\tilde{D}$ is a direct integral over the Bohr compactification $\mathbb{R}^n_B$ of operators $\tilde{D}_x$ defined by:

$$\tilde{D}_x = \sum a_i(x+y) D^i_y + b(x+y).$$

So $\tilde{D}$ acts on $L^2(\mathbb{R}^n_B \times \mathbb{R}^n)$. The algebra $\mathcal{A}$ is the algebra $\mathrm{CAP}^\infty(\mathbb{R}^n)$ of smooth almost periodic functions on $\mathbb{R}^n$, and $\mathcal{M}$ is the von Neumann crossed product algebra $L^\infty(\mathbb{R}^n) \rtimes \mathbb{R}^n_{discrete}$ which is a II_∞ factor as

proved in [39].

Let $(\mathcal{A}, \mathcal{M}, D)$ be a $p-$summable von-Neumann spectral triple. When D is invertible, we define the non commutative integral of $T \in \mathcal{M}$ by the formula:

$$\fint T := \tau_\omega(T|D|^{-p}).$$

When D is not invertible, we replace for instance $|D|$ by $(D^2+1)^{1/2}$. Note that $|D|-(D^2+1)^{1/2}$ is bounded. For simplicity, we shall usually assume that D is invertible, see also the next section. We point out the following useful proposition.

Proposition 2. *The map $T \mapsto \fint T$ is a hypertrace on the algebra $\tilde{\mathcal{A}}$ generated by $\mathcal{A}$ and $[D, \mathcal{A}] = \{[D, a], a \in \mathcal{A}\}$, i.e.*

$$\fint AT = \fint TA, \quad \forall T \in \mathcal{M} \text{ and } A \in \tilde{\mathcal{A}}.$$

Proof. The algebra $\tilde{\mathcal{A}}$ lies in $\bigcap_{n\geq 0} Dom(\delta^n)$, where δ is as before the unbounded derivation $T \mapsto [|D|, T]$. We have for any $A \in \tilde{\mathcal{A}}$:

$$\begin{aligned} \fint AT - \fint TA &= \tau_\omega([A,T]|D|^{-p}) \\ &= -\tau_\omega(T[|D|^{-p}, A]). \end{aligned}$$

Assume first that p is an integer. Then $[|D|^{-p}, A] = \sum_{k=0}^{p-1} |D|^{-k}[|D|^{-1}, A]|D|^{-p+k+1}$ and $[|D|^{-1}, A] = -|D|^{-1}[|D|, A]|D|^{-1}$. Therefore we get:

$$-T[|D|^{-p}, A] = \sum_{k=0}^{p-1} T|D|^{-k-1}[|D|, A]|D|^{-p+k}.$$

On the other hand, for any $k \in \{0, \cdots, p-1\}$, the operator $T|D|^{-k-1}[|D|, A]|D|^{-p+k}$ is trace-class since T and $[|D|, A]$ are bounded operators, and $|D|^{-p-1}$ is trace class. Therefore

$$\tau_\omega(T[|D|^{-p}, A]) = 0.$$

The proof is thus complete when p is an integer. Now if $p \notin \mathbb{N}$, we choose an integer k and a real number $r \in]0, 1[$ such that $p = rk$. Then an easy computation shows that we have:

$$[|D|^{-p}, A] = -\sum_{m=1}^{k} |D|^{-rm}[|D|^r, A]|D|^{-r(k-m+1)}.$$

Therefore, it suffices to show that $\tau_\omega(||D|^{-\alpha}S|D|^{-\beta}|) = 0$, where $\alpha = rm$, $\beta = r(k-m+1)$ and $S = [|D|^r, A]$. But the operator S is bounded. Indeed, one can for instance use the integral expression of $|D|^r$ given by:

$$|D|^r = C\int_0^{+\infty} |D|(tI + |D|)^{-1}t^{r-1}dt,$$

to deduce that for $t \leq 1$, the operator $|D|(tI + |D|)^{-1}$ is bounded with norm ≤ 1, while for $t \geq 1$, one can use the relation

$$[|D|(tI+|D|)^{-1}, A] = [|D|, A](tI + |D|)^{-1} - |D|(tI + |D|)^{-1}[|D|, A](tI + |D|)^{-1},$$

to conclude that S is bounded as allowed.

Now the end of the proof goes as follows. By [26][Theorem 4.2, p. 286], we have:

$$\int_0^t \mu_s(|D|^{-\alpha}S|D|^{-\beta})ds \leq \|S\| \int_0^t \mu_s(|D|^{-(\alpha+\beta)})ds,$$

and hence:

$$\tau_\omega(||D|^{-\alpha}S|D|^{-\beta}|) \leq \|S\|\tau_\omega(|D|^{-(p+r)}) = 0.$$

The last equality is a consequence of the summability of the operator $|D|^{-(p+r)}$. ∎

We point out that for a given $p-$summable von-Neumann spectral triple $(\mathcal{A},\mathcal{M},D)$ with $p>1$ and D invertible, we have by Theorem 1:

$$\begin{aligned}
⨍ T &= \lim_{t\to\omega}\frac{1}{\operatorname{Log}(t+1)}\tau(T1_{]\mu_t(|D|^{-1}),+\infty[}(|D|^{-1})|D|^{-p}) \\
&= \lim_{t\to\omega}\frac{1}{\operatorname{Log}(t+1)}\tau(T1_{]t^{-1/p},+\infty[}(|D|^{-1})|D|^{-p}),
\end{aligned}$$

for any $T\in\mathcal{M}$.

2.2. **Regularization formulae.** Let us give regularization formulae for the non commutative integral $⨍ T$. To this end, we need the following lemma.

Lemma 1. *Let $(\mathcal{A},\mathcal{M},D)$ be a $(p,\infty)-$summable von Neumann spectral triple with $p>1$ and D invertible. Let $f:[0,+\infty)\to\mathbb{C}$ be a C^1 function such that $\lim_{t\to+\infty}f(t)=0$ and $\int_1^{+\infty}\operatorname{Log}(t)|f'(t)|dt<+\infty$. Then, we have for any $T\in\mathcal{M}$:*

$$f(0)⨍ T=\lim_{\lambda\to\omega}\frac{1}{\operatorname{Log}(\lambda+1)}\tau(Tf(\frac{|D|}{\lambda^{1/p}})|D|^{-p}).$$

Proof. Let us first prove that the function

$$\lambda\longmapsto\frac{1}{\operatorname{Log}(\lambda+1)}\tau(Tf(\frac{|D|}{\lambda^{1/p}})|D|^{-p}),$$

is well defined and bounded. Since we have for any $x\geq 0$,

$$|f(x)|\leq\int_0^{+\infty}1_{[0,t[}(x)|f'(t)|dt,$$

we deduce that:

$$|f(\frac{|D|}{\lambda^{1/p}})|D|^{-p})|\leq\int_0^{+\infty}1_{]\frac{1}{\lambda t^p},+\infty[}(|D|^{-p})|D|^{-p}|f'(t)|dt. \tag{6}$$

There exists a constant $C>0$ such that $\mu_s(|D|^{-p})\leq C/(s+1)$, therefore we have for any $t>0$:

$$\begin{aligned}
\tau(1_{]\frac{1}{\lambda t^p},+\infty[}(|D|^{-p})|D|^{-p}) &= \int_{\{s>0,\mu_s(|D|^{-p})>\frac{1}{\lambda t^p}\}}\mu_s(|D|^{-p})ds \\
&\leq \int_0^{C\lambda t^p}\frac{C}{s+1}ds \\
&= C\operatorname{Log}(C\lambda t^p+1).
\end{aligned}$$

We thus deduce from (6) that

$$\tau(|f(\frac{|D|}{\lambda^{1/p}})|D|^{-p})|)\leq C\int_0^{+\infty}\operatorname{Log}(C\lambda t^p+1)|f'(t)|dt,$$

and hence:

$$\frac{1}{\operatorname{Log}(\lambda+1)}\tau(Tf(\frac{|D|}{\lambda^{1/p}})|D|^{-p})\leq\|T\|C\int_0^{+\infty}\frac{\operatorname{Log}(C\lambda t^p+1)}{\operatorname{Log}(\lambda+1)}|f'(t)dt|\leq C'\|T\|.$$

Thus, $S_T(f):=\lim_{\lambda\to\omega}\frac{1}{\operatorname{Log}(\lambda+1)}\tau(Tf(\frac{|D|}{\lambda^{1/p}})|D|^{-p})$ makes sense for any $f\in C_c^\infty(\mathbb{R})$, and defines a linear form S_T on $C_c^\infty(\mathbb{R})$. To prove the corollary, we may assume w.l.o.g. that $T\geq 0$. In this case, S_T is a positive Radon measure with support in $[0,+\infty[$. Moreover, since $\lim_{\lambda\to\omega}h(a\lambda)=\lim_{\lambda\to\omega}h(\lambda)$, for any $a>0$, we have by Theorem 1:

$$\int_{[0,1[}dS_T=\lim_{\lambda\to\omega}\frac{1}{\operatorname{Log}(\lambda+1)}\tau(T1_{]\lambda^{-1/p},+\infty}(|D|^{-1})|D|^{-p})=\tau_\omega(T|D|^{-p}), \tag{7}$$

and the scale invariance of S_T implies that $\operatorname{Supp}(S_T)=\{0\}$. By (7) again, we get:

$$S_T=\tau_\omega(T|D|^{-p})\delta_0,$$

and the proof is complete.

$\square$

From the above theorem, we deduce the following regularized spectral formula for the noncommutative integral $⨍ T$:

Theorem 3. *Let $(\mathcal{A}, \mathcal{M}, D)$ be a $(p,\infty)-$summable von Neumann spectral triple with $p>1$ and D invertible. Let $f:[0,+\infty)\to\mathbb{R}$ be a continuous polynomially decreasing function. Then, for any $T\in\mathcal{M}$, the function $\lambda\mapsto\frac{1}{\lambda}\tau(Tf(\lambda^{-1/p}|D|))$ is well defined and bounded for λ large enough, and we have:*

$$C_p(f) ⨍ T = \lim_{\lambda\to\omega}\frac{\tau(Tf(\frac{|D|}{\lambda^{1/p}})}{\lambda},$$

where $C_p(f) = p\int_0^{+\infty} f(t)t^{p-1}dt$.

This formula was proved by A. Connes [16] for $\mathcal{M} = B(\mathcal{H})$.

Proof. Let g, h be the functions defined on $[0,+\infty[$ by:

$$g(t) = t^p f(t) \text{ and } h(t) = \int_t^{+\infty} g(s)\frac{ds}{s} = \int_t^{+\infty} f(s)s^{p-1}ds.$$

These functions are then continuous on $[0,+\infty[$ and vanish at infinity. Moreover, $h'(t) = -f(t)t^{p-1}$ so that h is a C^1 function satisfying $\int_1^{+\infty}|h'(t)|\operatorname{Log}(t)dt<+\infty$.

Let us first prove that the function $\lambda\mapsto\frac{\tau(Tf(\frac{|D|}{\lambda^{1/p}})}{\lambda}$ is well defined and bounded for $\lambda>0$. We set for any $t>0$, $\varphi(t) = f(1/t)$. We thus define a non decreasing continuous function such that $\varphi(0)=0$, and hence:

$$\tag{8} \frac{\tau(f(\frac{|D|}{\lambda^{1/p}}))}{\lambda} = \frac{\tau(\varphi(\lambda^{1/p}|D|^{-1}))}{\lambda} = \frac{1}{\lambda}\int_0^{+\infty}\varphi(\lambda^{1/p}\mu_s(|D|^{-1}))ds.$$

But there exists a constant $C>0$ such that $\mu_s(|D|^{-1})\le\frac{C}{(s+1)^{1/p}}$, and since φ is non decreasing, we get:

$$\frac{\tau(f(\frac{|D|}{\lambda^{1/p}}))}{\lambda}\le\frac{1}{\lambda}\int_0^{+\infty}\varphi\left(C\left(\frac{\lambda}{s+1}\right)^{1/p}\right)ds = p\int_{C^{-1}\lambda^{-1/p}}^{+\infty} f(t)t^{p-1}C^p dt,$$

where the last equality follows from the change of variable $\frac{1}{t} = C(\frac{\lambda}{s+1})^{1/p}$. We deduce that $\lambda\mapsto\frac{\tau(f(\frac{|D|}{\lambda^{1/p}})}{\lambda}$ is bounded, and the inequality

$$|\frac{\tau(Tf(\frac{|D|}{\lambda^{1/p}}))}{\lambda}|\le\|T\|\frac{\tau(f(\frac{|D|}{\lambda^{1/p}}))}{\lambda}$$

shows that the function $\lambda\mapsto\frac{\tau(Tf(\frac{|D|}{\lambda^{1/p}}))}{\lambda}$ is well defined and bounded. On the other hand we have for any $x\ge 0$:

$$\frac{p}{\operatorname{Log}(\lambda)}\left[h(x/\lambda^{1/p})x^{-p} - h(x)x^{-p}\right] = \frac{1}{\operatorname{Log}(\lambda^{1/p})}\int_1^{\lambda^{1/p}} g(x/s)x^{-p}\frac{ds}{s} = \frac{1}{\operatorname{Log}(\lambda)}\int_1^{\lambda}\frac{1}{t}f\left(\frac{x}{t^{1/p}}\right)\frac{dt}{t},$$

and hence

$$\frac{p}{\operatorname{Log}(\lambda)}\left[Th\left(\frac{|D|}{\lambda^{1/p}}\right)|D|^{-p} - Th(|D|)|D|^{-p}\right] = \frac{1}{\operatorname{Log}(\lambda)}\int_1^{\lambda}\frac{1}{t}Tf(\frac{|D|}{t^{1/p}})\frac{dt}{t}.$$

By using the equality 8, it is easy to check that the right hand side is a Bochner integral of continuous functions with values in $L^1(\mathcal{M},\tau)$. On the other hand, τ is a continuous form on $L^1(\mathcal{M},\tau)$ and so:

$$\frac{p}{\operatorname{Log}(\lambda)}\left[\tau(Th\left(\frac{|D|}{\lambda^{1/p}}\right)|D|^{-p}) - \tau(Th(|D|)|D|^{-p})\right] = \frac{1}{\operatorname{Log}(\lambda)}\int_1^{\lambda}\theta(t)\frac{dt}{t},$$

with $\theta(t) = \frac{1}{t}\tau(Tf(\frac{|D|}{t^{1/p}})|D|^{-p}))$. Since θ is bounded, we get:

$$p \lim_{\lambda\to\omega} \frac{\tau(Th(\frac{|D|}{\lambda^{1/p}})|D|^{-p})}{\mathrm{Log}(\lambda)} = \lim_{\lambda\to\omega} M(\theta)(\lambda) = \lim_{\lambda\to\omega} \theta(\lambda).$$

Now by Lemma 1, the left hand side is equal to

$$ph(0)\tau_\omega(T|D|^{-p}) = p\left(\int_0^{+\infty} f(t)t^{p-1}dt\right)\tau_\omega(T|D|^{-p}).$$

We thus get:

$$C_p(f)\tau_\omega(T|D|^{-p}) = \lim_{\lambda\to\omega} \frac{1}{\lambda}\tau(Tf(\frac{|D|}{\lambda^{1/p}})),$$

and the proof is complete. □

Question. Is the above theorem true for a limiting process $\omega(f) = \lim_{t\to\omega} f(t)$ satisfying only the scale invariance property?

We can now deduce the Weil formula:

Corollary 1. *Let $(\mathcal{A}, \mathcal{M}, D)$ be a $p-$summable von Neumann spectral triple with $p > 1$ and D invertible. For any $T \in \mathcal{M}$, we have:*

$$\lim_{t^{-p}\to\omega} \left(t^p\tau(Te^{-t^2D^2})\right) = \Gamma(\frac{p}{2}+1)\int\!\!\!\!- T.$$

Proof. Take $f(x) = e^{-x^2}$ in the previous theorem. □

2.3. **Index theory in von Neumann algebras.** As before, let $\mathcal{M}$ be a von Neumann algebra in a Hilbert space H, equipped with a semi-finite normal faithful trace τ.

Lemma 2. *For any τ-compact projection $e \in \mathcal{M}$, we have $\tau(e) < +\infty$.*

Proof. Since $e = e^* = e^2$, we have $\mu_t(e) \in \{0, 1\}$. But $\mu_t(e) \to 0$ as $t \to +\infty$ by hypothesis, so that there exists t_0 such that

$$\mu_t(e) = 0, \text{ for } t \geq t_0,$$

and hence $\tau(e) = \int_0^{t_0} \mu_t(e)dt < +\infty$. □

Definition 6. An operator $T \in \mathcal{M}$ is called τ-Fredholm if there exists $S \in \mathcal{M}$ such that $1-ST$ and $1-TS$ are τ-compact.

Proposition 3. *If $T \in \mathcal{M}$ is τ-Fredholm, then the kernel and cokernel projections p_T and p_{T^*} are τ-finite.*

Proof. Let S be as in Definition 6. The projections $p_T = (1-ST)p_T$ and $p_{T^*} = (1-TS)^*p_{T^*}$ are τ-compact, and Lemma 2 gives the result. □

Definition 7. The index $\mathrm{Ind}_\tau(T)$ of a τ-Fredholm operator T is defined by:

$$\mathrm{Ind}_\tau(T) := \tau(p_T) - \tau(p_{T^*}), \tag{9}$$

where p_T and p_{T^*} are the projections on the kernel of T and T^* respectively.

Proposition 4. *If T and S are τ-Fredholm operators, then ST is a τ-Fredholm operator and*

$$\mathrm{Ind}_\tau(ST) = \mathrm{Ind}_\tau(T) + \mathrm{Ind}_\tau(S);$$

Proof. If T' and S' are parametrices for T and S respectively, then $T'S'$ is a parametrix for ST. So the composite of two τ-Fredholm operators is a τ-Fredholm operator. In addition:

$$\begin{aligned}\mathrm{Ind}_\tau(ST) = \tau(p_{ST}) - \tau(p_{T^*S^*}) &= \tau(p_T) + \tau(pr_{\mathrm{Ker}(ST)\ominus\mathrm{Ker}(T)}) - \tau(p_{S^*}) - \tau(pr_{\mathrm{Ker}(T^*S^*)\ominus\mathrm{Ker}(S^*)})\\ &= \tau(p_T) + \tau(pr_{\overline{Im(T)}\cap\mathrm{Ker}(S)}) - \tau(p_{S^*}) - \tau(pr_{\overline{Im(S^*)}\cap\mathrm{Ker}(T^*)}) = \mathrm{Ind}_\tau(T) + \mathrm{Ind}_\tau(S).\end{aligned}$$

□

Let us mention a technical lemma which will be used in the sequel.

Lemma 3. *Let $\mathcal{M}$ be a semi-finite von Neumann algebra acting on a Hilbert space H, and let τ be a (positive) normal faithful semi-finite trace on $\mathcal{M}$. Let e, f be two (orthogonal) projections in $\mathcal{M}$ and $A, B \in L^1(\mathcal{M}, \tau)$. Assume that $A \in e\mathcal{M}e$, $B \in f\mathcal{M}f$ and that there exists $V \in f\mathcal{M}e$ such that:*
(i) $VA = BV$;
(ii) $V : e(H) \to f(H)$ is injective with dense range.
Then $\tau(A) = \tau(B)$.

Proof. For any $\epsilon > 0$, let p_ϵ be the spectral projection of $|V|$ corresponding to the interval $(\epsilon, +\infty)$. Since we have $\|Vp_\epsilon x\| \leq \|Vx\|$, for any $x \in H$, the map $Vx \mapsto Vp_\epsilon x$ extends to a contraction $q_\epsilon \in B(f(H))$. Set $q_\epsilon x = 0$ for $x \in (1-f)(H)$. We thus define an operator $q_\epsilon \in f\mathcal{M}f$ which satisfies by construction:

$$q_\epsilon V = Vp_\epsilon.$$

From this relation, we deduce that $q_\epsilon^2 V = q_\epsilon V$ and hence $q_\epsilon^2 = q_\epsilon$ on $f(H)$. It follows that $q_\epsilon^2 = q_\epsilon$ and, since q_ϵ is a contraction, it is an orthogonal projection in $\mathcal{M}$ such that $q_\epsilon \leq f$. Moreover, we have:

$$q_\epsilon Bq_\epsilon V = q_\epsilon BVp_\epsilon = q_\epsilon VAp_\epsilon = Vp_\epsilon Ap_\epsilon.$$

From the inequality:

$$\|Vx\|^2 =< |V|^2 p_\epsilon x, p_\epsilon x > \;\geq \epsilon^2 \|p_\epsilon x\|^2 = \epsilon^2 \|x\|^2,$$

for any $x \in p_\epsilon(H)$, we get the existence of an inverse $W : q_\epsilon(H) \to p_\epsilon(H)$ for $V : p_\epsilon(H) \to q_\epsilon(H)$, such that $W \in p_\epsilon \mathcal{M} q_\epsilon$.

We have $Wq_\epsilon Bq_\epsilon V = p_\epsilon Ap_\epsilon$ and hence, since $B \in L^1(\mathcal{M}, \tau)$:

$$\tau(q_\epsilon Bq_\epsilon) = \tau(Wq_\epsilon Bq_\epsilon V) = \tau(p_\epsilon Ap_\epsilon). \tag{10}$$

Since V is injective, $p_\epsilon \to e$ strongly when $\epsilon \to 0$, and it follows from the relation $q_\epsilon V = Vp_\epsilon$ that $q_\epsilon \to f$ strongly when $\epsilon \to 0$. By the Lebesgue dominated convergence theorem in $L^1(\mathcal{M}, \tau)$, we deduce from Equation (10) that:

$$\tau(eAe) = \tau(fBf),$$

and finally $\tau(A) = \tau(B)$. The proof is complete. □

The following proposition generalizes the Calderon formula and computes $\mathrm{Ind}_\tau(T)$ by using the powers of $1 - ST$ and $1 - TS$. This formula will be used to get a polynomial expression for the Chern-Connes character (see 2.4).

Proposition 5. *Let $\mathcal{M}$ be a semi-finite von Neumann algebra with a (positive) normal faithful semi-finite trace τ and $T \in \mathcal{M}$. Assume that there exists $p \geq 1$ and an operator $S \in \mathcal{M}$ such that:*

$$1 - ST \in L^p(\mathcal{M}, \tau) \text{ and } 1 - TS \in L^p(\mathcal{M}, \tau).$$

Then T is τ-Fredholm and we have for any integer $n \geq p$:

$$\mathrm{Ind}_\tau(T) = \tau\left[(1 - ST)^n\right] - \tau\left[(1 - TS)^n\right]. \tag{11}$$

Proof. The operator T is τ-Fredholm because $L^p(\mathcal{M}, \tau) \cap \mathcal{M} \subset \mathcal{K}(\mathcal{M}, \tau)$. To prove the proposition, we may assume that $n = 1$. Indeed, let $S \in \mathcal{M}$ be such that:

$$A = 1 - ST \text{ and } B = 1 - TS$$

are in $L^p(\mathcal{M}, \tau) \cap \mathcal{M}$ (and hence in $L^n(\mathcal{M}, \tau)$ for any $n \geq p$) and set:

$$S' = S(1 + B + B^2 + ... + B^{n-1}).$$

We have:

$$1 - TS' = B^n \text{ and } 1 - S'T = A^n,$$

where the first relation is immediate and the second one uses the equality $TA = BT$. Replacing S by S', A by $A^n \in L^1(\mathcal{M}, \tau)$ and B by $B^n \in L^1(\mathcal{M}, \tau)$, we are thus reduced to the case where $n = 1$.

When $A = 1-ST$ and $B = 1-TS$ are in $L^1(\mathcal{M},\tau)$, we get from the relations $Ap_T = p_T$ and $p_{T^*}B = p_{T^*}$ the equality:

$$\mathrm{Ind}_\tau(T) = \tau(Ap_T) - \tau(p_{T^*}B).$$

To prove that $\mathrm{Ind}_\tau(T) = \tau(A) - \tau(B)$, it thus suffices to show that:

$$\tau(eAe) = \tau(fBf),$$

where $e = 1 - p_T$ and $f = 1 - p_{T^*}$. To this end, set $V := fTe$. We clearly have:

$$TeA = TA = BT = BfT,$$

and hence:

$$V(eAe) = (fBf)V.$$

If the intertwining operator V from $e(H)$ to $f(H)$ were invertible (the inverse would then be automatically in $e\mathcal{M}f$ by the bicommutant von Neumann theorem), we would get by cyclicity of the trace:

$$\tau(eAe) = \tau(fBf), \tag{12}$$

and Calderon's formula would be proved.

Although V is not necessarily invertible here, it is injective with dense range from $e(H)$ to $f(H)$. It turns out that this is enough to prove (12), by Lemma 3, and therefore Proposition 5 is proved. ☐

2.4. **The index map associated with a spectral triple.** In this subsection, we describe the index map $\mathrm{Ind}_{D,\tau}$ associated with a p-dimensional von Neumann spectral triple $(\mathcal{A},\mathcal{M},D)$. As usually, we shall replace D by $sgn(D)$, the sign of D. Let

$$D = F_1|D|$$

be the polar decomposition of the self-adjoint operator D. To get rid of the possible non injectivity of F_1 and following [15], we replace the Hilbert space $H = \mathrm{Ker}(D)^\perp \oplus \mathrm{Ker}(D)$ by

$$\mathcal{H} = H \oplus \mathrm{Ker}(D) \simeq H_1 \oplus H_2 \oplus H_3, \tag{13}$$

where $H_1 = \mathrm{Ker}(D)^\perp$, $H_2 = \mathrm{Ker}(D)$ and H_3 is an extra copy of $\mathrm{Ker}(D)$. Denote by e_1, e_2 and e_3 the projections onto H_1, H_2 and H_3 respectively. According to the splitting (13) of $\mathcal{H}$, set:

$$F = \begin{pmatrix} F_1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = F_1 + V \text{ where } V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

We thus define F, V in the semi-finite von Neumann algebra

$$\tilde{\mathcal{M}} = \mathcal{M} \oplus p_D\mathcal{M}p_D \subset B(\mathcal{H}),$$

which is equipped with the trace $\tilde\tau = \tau \oplus \tau$. Finally, embed $\mathcal{A}$ in $\tilde{\mathcal{M}}$ by $a \to a \oplus 0$. We have:

Lemma 4.
(1) $F = F^*$ and $F^2 = 1$;
(2) $\forall a \in \mathcal{A}, [F,a] \in L^{p,\infty}(\tilde{\mathcal{M}},\tilde\tau)$;
(3) $\forall a \in \mathcal{A}, aFa = aF_1a$;
(4) $\forall a \in \mathcal{A}$,$a(F - F_1) \in L^{p,\infty}(\tilde{\mathcal{M}},\tilde\tau)$.

Proof.
(1) Trivial.
(2) Note first that ap_D (and hence p_Da) belongs to $L^{p,\infty}(\mathcal{M},\tau)$, for any $a \in \mathcal{A}$. Indeed, we have:

$$ap_D = a(D+i)^{-1}(D+i)p_D = ia(D+i)^{-1}p_D \in L^{p,\infty}(\mathcal{M},\tau).$$

On the other hand we have $[F_1,a] \in L^{p,\infty}(\mathcal{M},\tau)$ for any $a \in \mathcal{A}$. Indeed we get by easy computations:

$$\begin{aligned}[F_1,a] &= [F_1,a](F_1+p_D)(F_1+p_D) \\ &= [F_1,a]F_1(F_1+p_D) + [F_1,a]p_D(F_1+p_D) \\ &= [F_1,a]F_1(F_1+p_D) + F_1ap_D(F_1+p_D),\end{aligned}$$

where $F_1 a p_D(F_1 + p_D) \in L^{p,\infty}(\mathcal{M}, \tau)$ by the previous observation. Moreover, we have:

$$[F_1, a]F_1 = [D, a](D+i)^{-1} + i[F_1, a]F_1(D+i)^{-1} - F_1[|D|, a](D+i)^{-1},$$

and hence $[F_1, a]F_1 \in L^{p,\infty}(\mathcal{M}, \tau)$. It follows that $[F_1, a]F_1(F_1 + p_D) \in L^{p,\infty}(\mathcal{M}, \tau)$ and finally that:

$$[F_1, a] \in L^{p,\infty}(\mathcal{M}, \tau) \quad \text{for any } a \in \mathcal{A},$$

as allowed. But we have:

$$[F, a] = [F_1, a] + [V, a], \text{ with } [V, a] = V p_D a(1 - e_3) - (1 - e_3) a p_D V e_3.$$

Since $[F_1, a], a p_D$ and $p_D a$ are in $L^{p,\infty}(\mathcal{M}, \tau)$, we finally get:

$$[F, a] \in L^{p,\infty}(\tilde{\mathcal{M}}, \tilde{\tau}),$$

and (2) is proved.

(3) Obvious.

(4) We have $a(F - F_1) = aV$ and hence $aV(aV)^* = aVV^*a^* = a p_D a$. It follows that:

$$\mu_s(aV) = \mu_s((aV)(aV)^*)^{1/2} = \mu_s((a p_D)(a p_D)^*)^{1/2} = \mu_s(a p_D), \text{ for any } s > 0,$$

and the result follows since we know that $a p_D \in L^{p,\infty}(\mathcal{M}, \tau)$. □

We are now in position to define the index map

$$\mathrm{Ind}_{D,\tau} : K_*(\mathcal{A}) \longrightarrow \mathbb{R},$$

associated with any von Neumann spectral triple $(\mathcal{A}, \mathcal{M}, D)$.

The even case. Assume that the spectral $(\mathcal{A}, \mathcal{M}, D)$ is even and denote by $\gamma \in \mathcal{M}$ the grading involution on $\mathcal{M}$. For any self-adjoint idempotent $e \in M_n(\mathcal{A})$, the operator:

$$T = e \circ (F \otimes 1_n) \circ e = e \circ (F_1 \otimes 1_n) \circ e$$

anticommutes with γ and satisfies:

$$T^2 - e = e \circ [F \otimes 1_n, e] \circ (F \otimes 1_n) \circ e = e \circ [F \otimes 1_n, e] \circ [F \otimes 1_n, e]. \tag{14}$$

It follows that $T^2 - e \in L^{p/2,\infty}(e(\tilde{\mathcal{M}} \otimes M_n(\mathbb{C}))e, \tilde{\tau} \otimes \mathrm{Tr})$ and hence T is a $(\tilde{\tau} \otimes \mathrm{Tr})$-Fredholm operator in the von Neumann algebra $e(\tilde{\mathcal{M}} \otimes M_n(\mathbb{C}))e$ acting on $e(\mathcal{H}^n)$.

Denote by $\mathrm{Ind}_{D,\tau}(e)$ the $(\tilde{\tau} \otimes \mathrm{Tr})$-index of the positive part of T acting from $e(\mathcal{H}^n_+)$ to $e(\mathcal{H}^n_-)$.

If e, e' are two self-adjoint idempotents representing a class $[e] - [e']$ in $K_0(\mathcal{A})$, then the number $\mathrm{Ind}_{D,\tau}(e) - \mathrm{Ind}_{D,\tau}(e')$ only depends on the class of $[e] - [e']$ in the even $K-$theory group $K_0(\mathcal{A})$ of the algebra $\mathcal{A}$. The τ-index map thus induces an additive map:

$$\mathrm{Ind}_{D,\tau} : K_0(\mathcal{A}) \longrightarrow \mathbb{R}.$$

The odd case. The construction of $\mathrm{Ind}_{D,\tau}$ for an odd von Neumann spectral triple $(\mathcal{A}, \mathcal{M}, D)$ is closely related to the Atiyah-Lusztig spectral flow in the context of von Neumann algebras [11]. Assume for simplicity that $\mathcal{A}$ is unital and let $P = (1 + F)/2$ be the Szego projection associated with the symmetry F. Consider for any invertible matrix $u \in GL_N(\mathcal{A})$, the Toeplitz operator:

$$T := (P \otimes 1_N) \circ u \circ (P \otimes 1_N).$$

Since we have:

$$\begin{aligned}(P \otimes 1_N) \circ u^{-1} \circ (P \otimes 1_N) \circ u \circ (P \otimes 1_N) - (P \otimes 1_N) = \\ (1/4)[F \otimes 1_N, u^{-1}] \circ [F \otimes 1_N, u] \circ (P \otimes 1_N) \in L^{p/2,\infty}(\tilde{\mathcal{M}} \otimes \mathrm{End}(\mathbb{C}^N)),\end{aligned} \tag{15}$$

and

$$\begin{aligned}(P \otimes 1_N) \circ u \circ (P \otimes 1_N) \circ u^{-1} \circ (P \otimes 1_N) - (P \otimes 1_N) = \\ (1/4)[F \otimes 1_N, u] \circ [F \otimes 1_N, u^{-1}] \circ (P \otimes 1_N),\end{aligned} \tag{16}$$

we deduce that T is a $(\tau\otimes \mathrm{Tr})$-Fredholm operator. Denote by $\mathrm{Ind}_{D,\tau}(u)$ the $(\tau\otimes\mathrm{Tr})$-index of T. By classical arguments, we again easily see that $\mathrm{Ind}_{D,\tau}(u)$ only depends on the class of u in the odd $K-$theory group $K_1(\mathcal{A})$ of the algebra $\mathcal{A}$. We get in this way an additive map:

$$\mathrm{Ind}_{D,\tau} : K_1(\mathcal{A}) \longrightarrow \mathbb{R}.$$

To sum up, any von Neumann spectral triple $(\mathcal{A},\mathcal{M},D)$ gives arise to an index map:

$$\mathrm{Ind}_{D,\tau} : K_*(\mathcal{A}) \longrightarrow \mathbb{R}.$$

This map will be described in Theorem 4 as a pairing with a (polynomial) cyclic cocycle on the algebra $\mathcal{A}$.

We shall use the cyclic cohomology of the algebra $\mathcal{A}$ and we proceed now to recall the main definitions for the convenience of the reader. Our main references are [17, 15, 31, 41].

Let $\mathcal{A}$ be a topological algebra. Denote for $k\geq 0$ by $\mathcal{C}^k(\mathcal{A})$ the vector space of jointly continuous $(k+1)-$linear forms on $\tilde{\mathcal{A}}\times\mathcal{A}^{k+1}$, where $\tilde{\mathcal{A}}$ is the (minimal) unitalization of $\mathcal{A}$ even if $\mathcal{A}$ is already unital. The elements of $\mathcal{C}^k(\mathcal{A})$ are called (continuous) Hochschild cochains on $\mathcal{A}$. The Hochschild coboundary $b:\mathcal{C}^k(\mathcal{A})\to\mathcal{C}^{k+1}(\mathcal{A})$ is the differential defined for any $\varphi\in\mathcal{C}^k(\mathcal{A})$ by:

$$b\varphi(\tilde{a}_0,a_1,\cdots,a_{k+1}) := \sum_{j=1}^{k}(-1)^j\varphi(\tilde{a}_0,a_1,\cdots,a_ja_{j+1},a_{j+2},\cdots,a_{k+1})+$$
$$\varphi(\tilde{a}_0a_1,a_2,\cdots,a_{k+1})+(-1)^{k+1}\varphi(a_{k+1}\tilde{a}_0,a_1,\cdots,a_k).$$

We have $b^2=0$. The homology of the resulting complex $(\mathcal{C}^*(\mathcal{A}),b)$ is called the Hochschild cohomology of the algebra $\mathcal{A}$ and denoted $\mathrm{HH}^*(\mathcal{A})$. For instance, one can see that $0-$dimensional Hochschild cocycles are exactly continuous traces on $\mathcal{A}$. Cyclic cohomology is obtained by using a suitable subcomplex of the Hochschild complex. More precisely, we consider the subspace $\mathcal{C}^k_\lambda(\mathcal{A})$ of $\mathcal{C}^k(\mathcal{A})$ built up from jointly continuous $(k+1)-$linear forms on $\tilde{\mathcal{A}}\times\mathcal{A}^{k+1}$ which are equivariant with respect to the action of the cyclic group generated by the permutation $\lambda(0,1,\cdots,k)=(k,0,1,\cdots,k-1)$. So a Hochschild cochain φ is a cyclic cochain if

$$\varphi(a^k,a^0,\cdots,a^{k-1})=(-1)^k\varphi(a^0,\cdots,a^k),\quad \forall a^j\in\mathcal{A}.$$

The Hochschild differential b preserves the subspace $\mathcal{C}^*_\lambda(\mathcal{A})$ and we get a differential complex $(\mathcal{C}^*_\lambda(\mathcal{A}),b)$ called the cyclic complex of $\mathcal{A}$. Its homology is called the cyclic cohomology of $\mathcal{A}$ and is denoted $\mathrm{HC}^*(\mathcal{A})$ or equivalently $\mathrm{H}^*_\lambda(\mathcal{A})$.

The short exact sequence of complexes

$$0\to\mathcal{C}^*_\lambda(\mathcal{A})\hookrightarrow\mathcal{C}^*(\mathcal{A})\longrightarrow\mathcal{C}^*(\mathcal{A})/\mathcal{C}^*_\lambda(\mathcal{A})\to 0$$

induces the famous $(SBI)-$long exact sequence [17]

$$\cdots\xrightarrow{S}\mathrm{HC}^k(\mathcal{A})\xrightarrow{I}\mathrm{HH}^k(\mathcal{A})\xrightarrow{B}\mathrm{HC}^{k-1}(\mathcal{A})\xrightarrow{S}\mathrm{HC}^{k+1}(\mathcal{A})\xrightarrow{I}\cdots$$

The operator I is induced by the inclusion and the operator B is defined for instance in [17]. We have $B^2=0$ and $bB+Bb=0$. Using the bicomplex $(\mathcal{C}^{k,h}(\mathcal{A}),b,B)$ with $\mathcal{C}^{k,h}(\mathcal{A}):=\mathcal{C}^{k-h}(\mathcal{A})$ for $k\geq h\geq 0$, we actually recover the cyclic cohomology of $\mathcal{A}$, see for instance [31]. The operator $S:\mathrm{HC}^*\to\mathrm{HC}^{*+2}$ is Connes' periodic operator, see the definition in [17][page ?]. The homotopy invariants will rather live in periodic cyclic cohomology. This is defined as the strict indutive limit of cyclic cohomology with respect to the operator S and is denoted $\mathrm{HP}^*(\mathcal{A})$. Therefore, $\mathrm{HP}^*(\mathcal{A})$ is a $\mathbb{Z}_2-$graded theory with many topological properties similar to those of $K-$theory.

The main property of cyclic cohomology which will be used in the sequel is its pairing with $K-$theory. So cyclic cocycles furnish group morphisms from $K-$theory to the scalars. More precisely, if φ is, say, a

$(2k)-$cyclic cocycle on $\mathcal{A}$, then the following formula induces a pairing with the K_0-group of $\mathcal{A}$, see [17]:

$$<\varphi, e>:=\sum_{i_0,\cdots,i_{2k}=1}^{N}\varphi(e_{i_0i_1}, e_{i_1i_2},\cdots, e_{i_{2k}i_0}),\quad e\in M_N(\mathcal{A}), e^2=e.$$

In the same way, any odd cyclic cocycle induces a pairing with the K_1-theory and a similar explicit formula holds in the odd case.

Remark 2. The above definitions of the continuous cyclic cohomology can be generalized to algebras with topologies which are not necessarily topological algebras. Such algebras are then allowed to satisfy some weaker assumptions, see for instance [4, 5] or [21].

Let ϕ be a cyclic k-cocycle on the algebra $\mathcal{A}$. As in [17], we shall denote for any $N\geq 1$, by $\phi\sharp\,\mathrm{Tr}$ the cyclic k-cocycle on $M_N(\mathcal{A})$ given by:

$$(\phi\sharp\,\mathrm{Tr})(a^0\otimes A^0,\cdots, a^k\otimes A^k):=\phi(a^0,\cdots,a^k)\,\mathrm{Tr}(A^0\cdots A^k),$$

for any $(a^0,\cdots,a^k)\in\mathcal{A}^{k+1}$ and any $(A^0,\cdots,A^k)\in M_N(\mathbb{C})^{k+1}$.

Theorem 4. *Let $(\mathcal{A},\mathcal{M},D)$ be a von Neumann-spectral triple of dimension p and let F be the symmetry associated with D as above.*
(1) If $(\mathcal{A},\mathcal{M},D)$ is even with grading involution γ, the formula:

$$\phi_{2k}(a^0,...,a^{2k})=(-1)^k\tau(\gamma a^0[F,a^1]\cdots[F,a^{2k}]);$$

defines, for any $k>p/2$, a $2k$-cyclic cocycle on the algebra $\mathcal{A}$ and we have for any projection e in $M_N(\mathcal{A})$:

$$\mathrm{Ind}_{D,\tau}(e)=(\phi_{2k}\sharp\,\mathrm{Tr})(e,\cdots,e).$$

(2) If $(\mathcal{A},\mathcal{M},D)$ is odd, then for any $k>p/2$, we define a $2k+1$-cyclic cocycle on the algebra $\mathcal{A}$ by setting:

$$\phi_{2k+1}(a^0,...,a^{2k+1})=(-1/2^{2k+1})\tau(a^0[F,a^1]...[F,a^{2k+1}]);$$

Assume that $\mathcal{A}$ is unital, then for any invertible u in $M_N(\mathcal{A})$, we have:

$$\mathrm{Ind}_{D,\tau}(u)=(\phi_{2k+1}\sharp\,\mathrm{Tr})(u^{-1},u,...,u^{-1},u).$$

Proof. The proof follows the lines of [17].

(1) We first point out that ϕ_{2k} is in evidence a cyclic cochain. Let $(a^0,\cdots,a^{2k+1})\in\mathcal{A}^{2k+2}$. Then we have:

$$b(\phi_{2k})(a^0,\cdots,a^{2k+1})=(-1)^k\tau(\gamma[a^0[F,a^1]\cdots[F,a^{2k}],a^{2k+1}])$$
$$=(-1)^k\tau([a^0[F,a^1]\cdots[F,a^{2k}],\gamma a^{2k+1}])=0.$$

Therefore the cochain ϕ_{2k} is a Hochschild cocycle on $\mathcal{A}$.

From Equation (14), we deduce that the operator $T=(e\circ(F\otimes id_N)\circ e)_+$ is τ-Fredholm in $e(\mathcal{M}\otimes M_N(\mathbb{C}))e$ with parametrix given by $S=(e\circ(F\otimes id_N)\circ e)_-$. Moreover, $e-ST$ as well as $e-TS$ are in $L^k(e(\mathcal{M}\otimes M_N(\mathbb{C}))e,\tau\otimes\mathrm{Tr})$. Therefore Proposition 5 gives:

$$\mathrm{Ind}_\tau((eFe)_+)=(\tau\sharp\,\mathrm{Tr})(\gamma\circ(e-(e\circ(F\otimes id_N)\circ e)^2)^k).$$

Computing $(e-(e\circ(F\otimes id_N)\circ e)^2)^k$ and using the relation $e\circ[F\otimes 1_N,e]\circ e=0$, we obtain:

$$(e-(e\circ(F\otimes id_N)\circ e)^2)^k=(-1)^k e\circ[F\otimes 1_N,e]^{2k},$$

and hence the conclusion.

(2) That ϕ_{2k+1} is cyclic is again obvious. Let $(a^0,\cdots,a^{2k+2})\in\mathcal{A}^{2k+3}$. Then we have:

$$b(\phi_{2k+1})(a^0,\cdots,a^{2k+2})=(-1/2^{2k+1})\tau([a^0[F,a^1]\cdots[F,a^{2k+1}],a^{2k+2}])=0.$$

Hence ϕ_{2k+1} is a cylic cocycle on $\mathcal{A}$.

To compute the τ-index of $P\circ u\circ P$, we again apply the Calderon formula. From the relations (15) and (16), we deduce that $T := P\circ u\circ P$ is τ-Fredholm in $P(\mathcal{M}\otimes M_N(\mathbb{C}))P$ with parametrix given by $S = P\circ u^{-1}\circ P$. Moreover, $P-ST$ and $P-TS$ are in $L^k(P(\mathcal{M}\otimes \operatorname{End}(\mathbb{C}^N))P, \tau\otimes \operatorname{Tr})$. Therefore Proposition 5 gives:

$$\operatorname{Ind}_\tau(P\circ u\circ P) = (\tau\sharp\operatorname{Tr})((P-(P\circ u^{-1}\circ P\circ u\circ P))^k) - (\tau\sharp\operatorname{Tr})((P-(P\circ u\circ P\circ u^{-1}\circ P))^k).$$

The computation of $P-(P\circ u^{-1}\circ P\circ u\circ P)$ in (15) and (16) gives:

$$P-(P\circ u^{-1}\circ P\circ u\circ P) = -[P,u^{-1}]\circ[P,u]\circ P \text{ and } P-(P\circ u\circ P\circ u^{-1}\circ P) = -[P,u]\circ[P,u^{-1}]\circ P.$$

But,

$$([P,u^{-1}]\circ[P,u]\circ P)^k = ([P,u^{-1}]\circ[P,u])^k\circ P \text{ and } ([P,u]\circ[P,u^{-1}]\circ P)^k = ([P,u]\circ[P,u^{-1}])^k\circ P.$$

On the other hand we have:

$$[P,u^{-1}]\circ[P,u] = -u^{-1}\circ[P,u]\circ u^{-1}\circ[P,u] \text{ and } [P,u]\circ[P,u^{-1}] = -u\circ[P,u^{-1}]\circ u\circ[P,u^{-1}],$$

Therefore,

$$([P,u^{-1}]\circ[P,u])^k = (-1)^k(u^{-1}\circ[P,u])^{2k} = (-1)^k u^{-1}\circ([P,u]\circ u^{-1})^{2k-1}\circ[P,u],$$

and a similar result holds for $([P,u]\circ[P,u^{-1}])^k$ and we get:

$$\operatorname{Ind}_\tau(P\circ u\circ P) = (-1)^k(\tau\sharp\operatorname{Tr})(P\circ u^{-1}\circ([P,u]\circ u^{-1})^{2k-1}\circ[P,u] - P\circ([P,u]\circ u^{-1})^{2k-1}\circ[P,u]\circ u^{-1}).$$

Hence we get using the trace property of τ:

$$\operatorname{Ind}_\tau(P\circ u\circ P) = (-1)^k(\tau\sharp\operatorname{Tr})\left([P,u^{-1}]\circ([P,u]\circ u^{-1})^{2k-1}\circ[P,u]\right),$$

and thus we finally obtain:

$$\operatorname{Ind}_\tau(P\circ u\circ P) = (-1/2^{2k+1})(\tau\sharp\operatorname{Tr})\left(u^{-1}\circ[F\otimes 1_N,u]\circ([F\otimes 1_N,u^{-1}]\circ[F\otimes 1_N,u])^k\right),$$

which completes the proof. □

Remark 3. One can define a cyclic cocycle of minimal order. In the even case for instance, there is a well defined cyclic $p-$cocycle that can be associated with the spectral triple in the following way:

$$\phi_p(a^0,\cdots,a^p) := \frac{(-1)^{p/2}}{2}\tau(\gamma F[F,a^0]\cdots[F,a^p]).$$

This cocycle also represents the index map associated with the spectral triple. The proof is an easy extension of the one given in [17].

Remark 4. Assume that G is a compact Lie group which acts on the even spectral triple [3]. So G acts by unitaries in $\mathcal{M}$, this action preserves $\mathcal{A}$ and the operator D is G-invariant. We denote by $U(g)$ the unitary corresponding to $g\in G$. Then the equivariant index $\operatorname{Ind}_\tau^G((eFe)_+)$ of $(eFe)_+$ does make sense as an element of $R(G)\otimes\mathbb{R}$, where $R(G)$ is the representation ring of G. We get using a similar proof the following equivariant polynomial index formula:

$$\forall g\in G, \operatorname{Ind}_\tau^G((eFe)_+)(g) = (\phi_{2k}\sharp\operatorname{Trace})(U(g)\circ e,e,...,e).$$

See [3]. A similar result holds in the odd case.

So associated with any von Neumann spectral triple, there is an index problem which can be stated as follows:

"Give a local formula for the traced index map

$$K_*(\mathcal{A})\longrightarrow\mathbb{R}."$$

Using Theorem 4, we see that the index problem can be stated in the cyclic cohomology world:

"Find a local cyclic cocycle ψ on $\mathcal{A}$ such that:

$$<\psi,x> = <\phi,x>,\quad \forall x\in K_*(\mathcal{A})."$$

Here ϕ is the cyclic cocycle defined in Theorem 4.

This index problem reduces to the index problem solved by A. Connes and H. Moscovici in [19] if one takes the usual von Neumann algebra of operators in a Hilbert space with the usual trace.

Examples. In the examples listed after Definition 5, the index problem becomes:

(1) In the first example of Riemannian geometry, we recover the classical index problem which was solved by Atiyah and Singer in [2, 15].

(2) In the case of measured foliations we recover the measured index problem which was solved by A. Connes in [13].

(3) In the case of Galois coverings, we recover the index problem which was solved by M.F Atiyah in [1].

(4) For almost periodic operators, we obtain the Shubin index problem that was solved in [39]. The index map yields here a morphism:

$$\mathrm{Ind}_{D,\tau} : K^0(\mathbb{R}^n_B) \longrightarrow \mathbb{R},$$

where $\mathbb{R}^n_B$ is the Bohr compactification of $\mathbb{R}^n$.

Up to normalizing constants, the sequence ϕ_n of Theorem 4 can be arranged to represent a periodic cyclic cocycle on $\mathcal{A}$ [17], i.e. up to appropriate constants, we have:

$$S(\phi_n) = \phi_{n+2},$$

where S is Connes' periodic operator. The periodic cyclic class obtained is called the Chern-Connes character of the von Neumann spectral triple. In [6] we give a local formula for this Chern-Connes character using residues of zeta functions and following the method of [19]. This local formula unifies all the examples listed above and gives a complete solution to the von Neumann index problem.

Remark 5. Any even von Neumann spectral triple gives rise for n large enough to a homomorphism from the Cuntz algebra $q\mathcal{A}$ of $\mathcal{A}$ to the algebra $\mathcal{J} = L^{2n}(\mathcal{M}, \tau)$ [18]. This shows that the Chern-Connes character of the von Neumann spectral triple can also be defined following the method of [18].

3. Measured foliations

Let (M, F) be a smooth foliated manifold with (for simplicity) even dimensional spin leaves. Denote by G the holonomy groupoid of (M, F). Let $\mathcal{S}$ be the hermitian spin vector bundle and D the $G-$operator constructed out of the Dirac operator along the leaves following [13]. For all the background material about $G-$operators, we refer to the seminal paper [13]. We fix a Lebesgue-class measure α on the leaf manifold $\mathcal{F}$ and the lifted Haar system $\nu = (\nu_x)_{x\in M}$ on G. We assume furthermore that there exists a positive holonomy invariant transverse measure Λ, then the data (Λ, α) enables to define a measure on the manifold M that we denote by Λ_ν [13]. We will denote by $W^*_\Lambda(M, F; \mathcal{S})$ the von Neumann algebra associated with Λ and $\mathcal{S}$ [6]. This von Neumann algebra is then endowed with a trace τ_Λ which turns out to be faithful by construction. Recall that $W^*_\Lambda(M, F; \mathcal{S})$ is represented in the Hilbert space $H = \int_M^\oplus L^2(G^x, s^*(\mathcal{S}), \nu^x) d\Lambda_\nu(x)$ of Λ_ν-square integrable sections of the field of Hilbert spaces $(H_x = L^2(G^x, s^*(\mathcal{S}), \nu^x))_{x\in M}$, where Λ_ν is the positive measure on M constructed out of Λ and ν as in [13].

Definition 8. Let (M, F, Λ) be a measured $p-$dimensional foliation on a compact manifold M. For any pseudodifferential G-operator P of order $-p$ acting on sections of a vector bundle E over M, we define the foliated local residue $\mathrm{res}(P) \in C^{\infty,0}(M, |\Lambda|^1 F)$ as the longitudinal 1-density, given locally by:

$$\mathrm{res}_{(u,t)}(P) = \frac{1}{(2\pi)^p}\left[\int_{\|\xi\|=1} \mathrm{tr}(\sigma_{-p}P((u,t),\xi))|d\xi|\right]|du|,$$

where $\sigma_{-p}P((u,t),\xi)$ is the principal symbol of P.

This local residue is well defined (see for instance [30], p.17) and we have the following generalization of the well known result in the non foliated case and which shows the locality of the von Neumann Dixmier

trace in the case of measured foliations.

Theorem 5. *Let (M, F, Λ) be a measured p-dimensional foliation on a compact manifold M. Let P be a pseudodifferential $G-$operator of order $-p$ acting on sections of a vector bundle E over M, and denote by $\sigma_{-p}(P)$ its principal symbol. Then we have*

(i) P belongs to the Dixmier ideal $L^{1,\infty}(W^*_\Lambda(M, F; E), \tau_\Lambda)$ associated with the von-Neumann algebra $W^*_\Lambda(M, F; E)$ and its trace τ_Λ;

(ii) For any invariant mean ω, the Dixmier trace of P is given by

$$\tau^\Lambda_\omega(P) = \frac{1}{p}\int_{M/F} \operatorname{res}_L(P_L)d\Lambda(L),$$

where $\operatorname{res}_L(P_L)$ is the foliated local residue of Definition 8 and τ^Λ_ω is the Dixmier trace associated with τ_Λ and ω as in Section 1.

Proof. (i) By [15][page 126], we have

$$P = \sum_{1\leq i\leq k} P_i + R$$

where R is an infinitely smoothing G-operator and each $P_i \in \psi^{-p}(M, F; E)$ is given by a continuous familly with compact support in $\psi^{-p}_c(\Omega_i, E)$ with Ω_i a distinguished foliation chart trivializing E. Since R is trace-class with respect to τ^Λ, [13][Prop. 6.b, page 131] and $L^1(W^*_\Lambda(M, F; E)) \subset L^{1,\infty}(W^*_\Lambda(M, F; E))$ we may assume that $P \in \psi^{-p}_c(\Omega, E)$ where Ω is a distinguished foliation chart trivializing E. We thus may work locally assuming that

$$M = T^p \times D^{n-p}$$

is foliated by $T^p \times \{t\}$, for $t \in D^{n-p}$ and $P = (P_t)_{t\in D^{n-p}}$ is a continuous familly of scalar pseudodifferential operators of order $-p$ on T^p (the proof for matrices is the same). Here T^p is the standard p-torus and D^{n-p} is the unit disk in $\mathbb{R}^{n-p}$.

For any $t \in D^{n-p}$, let $\Delta_t = \Delta$ be the usual Laplacian on the flat torus T^p. Since $L^{1,\infty}(W^*_\Lambda(M, F))$ is an ideal in $W^*_\Lambda(M, F)$, we only have to show that the constant familly $(1 + \Delta_t)^{-p/2}$ defines an element in $L^{1,\infty}(W^*_\Lambda(M, F))$. Indeed we have

$$P_t = Q_t(1 + \Delta_t)^{-p/2}, \quad (t \in D^{n-p})$$

where $Q_t = P_t(1 + \Delta_t)^{p/2}$ is a continuous family of 0-order pseudodifferential operators on T^p, and hence defines an element of $W^*_\Lambda(M, F) \cong L^\infty(D^{n-p}, \Lambda) \otimes B(L^2T^p)$ by [13][page 126, Proposition 1.b].

Since we trivially have for any $T \in B(L^2(T^p))$:

$$\mu^{\tau^\Lambda}_{\Lambda(D^{n-p})s}(1 \otimes T) = \mu^{\mathrm{Tr}}_s(T),$$

we get:

$$\frac{1}{\operatorname{Log}(R)}\int_0^R \mu^{\tau^\Lambda}_s((1 \otimes (1 + \Delta))^{-p/2})ds = \frac{\Lambda(D^{n-p})}{\operatorname{Log}(R/\Lambda(D^{n-p}))}\int_0^{R/\Lambda(D^{n-p})} \mu_s(1 + \Delta)ds.$$

The right hand side of this equality converges to $\frac{\Lambda(D^{n-p})}{p} \times Area(T^p)$. Henceforth, $1 \otimes (1 + \Delta)^{-p/2}$ belongs to $L^{1,\infty}(W^*_\Lambda(M, F))$ and we get

$$\tau^\Lambda_\omega(1 \otimes (1 + \Delta)^{-p/2}) = \frac{\Lambda(D^{n-p})}{p} \times Area(T^p).$$

(ii) We may work locally and assume again that $M = T^p \times D^{n-p}$. For any smooth function $\sigma = \sigma(u, \xi, t) \in \mathbb{S}^*T^p \times D^{n-p}$, set

$$\nu(\sigma) = \tau^\Lambda_\omega(P),$$

where P is any classical tangential pseudodifferential operator of order $-p$ with principal symbol equal to σ. Since two classical pseudodifferential operators of order $-p$ with the same principal symbol coincide modulo $\psi^{-(p+1)}(M,F)$ and since

$$\psi^{-(p+1)}(M,F) \subset L^1(W^*_\Lambda(M,F)) \subset \mathrm{Ker}(\tau^\Lambda_\omega),$$

we deduce that $\nu(\sigma)$ is well defined. It is clear that ν is a positive linear form on $C^\infty(\mathbb{S}^*T^p \times D^{n-p})$ and is in fact a positive measure on $T^*(T^p)_1 \times D^{n-p}$.

Let $\nu = \int_{D^{n-p}} \nu_t d\rho(t)$ be the disintegration of ν with respect to the projection $\pi : \mathbb{S}^*(T^p) \times D^{n-p} \to D^{n-p}$ [8][page 58]. For any isometry g of T^p, the measure ν is invariant under the action of g on the fibers of π because τ^Λ_ω is a trace. By uniqueness of the disintegration of ν we get

$$g(\nu_t) = \nu_t, \quad \rho- \text{ a.e. in } t,$$

so that ν_t is proportional to the volume form on $\mathbb{S}^*T^p$ for almost every t. We thus have

$$\tau^\Lambda_\omega(P) = \int_{D^{n-p}} [\int_{\mathbb{S}^*T^p} \sigma_{-p}(P)(u,\xi,t) dv(u,\xi)] h(t) d\rho(t),$$

where h is a bounded ρ-measurable positive function on D^{n-p}. Let us prove now that the measure $hd\rho$ is proportional to $d\Lambda$.

For any continous function f on D^{n-p}, we know by (i) that the Dixmier trace of the continuous family $P_t = f(t)(1+\Delta_t)^{-p/2}$ where $\Delta = \Delta_t$ is now the Laplacian on the standard sphere, is given by

$$\tau^\Lambda_\omega(P) = C_1 \times \Lambda(f),$$

the constant C_1 being independent of f.

On the other hand, we have

$$\tau^\Lambda_\omega(P) = C_2 \times \int_{D^{n-p}} f(t)h(t)d\rho(t),$$

where the constant C_2 does no more depend on f. We thus get the existence of a constant $C > 0$ such that

$$hd\rho = Cd\Lambda.$$

It follows that

$$\tau^\Lambda_\omega(P) = C \int_{M/F} \mathrm{res}_L(P_L) d\Lambda(L),$$

and the computation of (i) shows that $C = 1/p$. □

Proposition 6. *Let γ be the grading induced by the grading of the spin bundle $\mathcal{S}$ ($\dim(F) = 2r$) and let $\mathcal{G}$ be the symmetry constructed out of D like in Section 2 so that $\mathcal{G}^2 = 1$. For any $f \in C^\infty(M)$, denote by $\pi(f)$ the 0 order differential G-operator defined by f, say multiplication by $f \circ s$ on each $L^2(G^x, \nu^x, s^*\mathcal{S})$. Then:*

(i) $(C^\infty(M), W^*(M,F;\mathcal{S}), D)$ is an even von-Neumann spectral triple of finite dimension equal to the dimension of the leaves;

(ii) $\forall k > r$,

$$\phi_k(f_0, f_1, ..., f_{2k}) = (-1)^k (\tau_\Lambda \sharp trace)(\gamma \circ \pi(f_0) \circ [\mathcal{G}, \pi(f_1)] \circ ... \circ [\mathcal{G}, \pi(f_{2k})])$$

defines a cyclic cocycle on the algebra $C^\infty(M)$;

(iii) Let $e \in M_N(C^\infty(M))$ be the projection corresponding to a stabilization of the complex vector bundle E. Then we have for any $k > r$:

$$\mathrm{Ind}_\Lambda([D_E]_+) = \phi_k(e, \cdots, e).$$

Proof. We only have to prove (i), the rest of the proposition being a rephrasing of Theorem 4 in the present situation. We first point out that D is affiliated with $W^*(M,F;\mathcal{S})$ and that $\forall f \in C^\infty(M), [D,f]$ is in $W^*(M,F;\mathcal{S})$ because it is affiliated and bounded. On the other hand the principal symbol of $|D|$ commutes with those of all order 0 pseudodifferential G-operators, so that (ii) and (iii) in Definition 5 are satisfied.

Let now $Q \in \psi^{-1}(M,F;\mathcal{S})$ be a parametrix for the elliptic G-operator D so that

$$1 - QD = R \text{ and } 1 - DQ = R'$$

are regularizing operators, say live in $C_c^{\infty,0}(G, \mathrm{End}(\mathcal{S}))$. The existence of Q is proved in [13]. Then we have

$$(D+i)^{-1} = Q + (D+i)^{-1}R'$$

so that $(D+i)^{-1}$ and Q are in the same Dixmier ideal. But $Q \in L^{2r,\infty}(W^*(M,F;\mathcal{S}), \tau_\Lambda)$ and the conclusion follows. □

For any smooth complex vector bundle E over M, the twisted Dirac operator D_E (lifted again to become a G-operator) is a well defined elliptic differential G-operator. The von Neumann index problem then asks for a computation of the measured analytic map $K^0(M) \to \mathbb{R}$ given by:

$$[E] \to \mathrm{Ind}_\Lambda([D_E]_+),$$

as a pairing of E with a cyclic cocycle on $C^\infty(M)$. Whence we joint the usual measured index problem, at least for spin foliations.

4. The local positive Hochschild class

In this final section, we shall prove a local formula for the image of the Chern-Connes character of a von Neumann spectral triple in Hochschild cohomology. More precisely, we shall give a *local* representative of this class in terms of the Dixmier trace associated with some state ω. The formula that we obtain shows at the same time the positivity of the Hochschild cocycle [18, 15]. Our results follow from the classical case treated by A. Connes in the unpublished paper [16] modulo the results of the previous sections. A proof of these technical results in the type I case also appeared in the meantime in [42].

For the sake of simplicity, we shall restrict ourselves to the even case. The main problem is the following. The Chern-Connes character $\mathrm{Ch}(\mathcal{A},\mathcal{M},D)$ of an even p-dimensional von Neumann spectral triple $(\mathcal{A},\mathcal{M},D)$ can be described in the $(b,B)-$bicomplex by a family $(\varphi_{2k})_{k\geq 0}$ such that $b\varphi_{2k} + B\varphi_{2k+2} = 0$. Then the pairing $< \mathrm{Ch}(\mathcal{A},\mathcal{M},D), [e] >$ with projections is given up to normalization by the formula [15][page 271]

$$< [\varphi], [e] >= \sum_{k\geq 0} (-1)^k \frac{(2k)!}{k!} \varphi_{2k}(e - \frac{1}{2}, e, \cdots, e).$$

A solution to the index problem is a precise periodic cyclic cocycle φ where each φ_{2k} is given by a formula which is local in the sense of Connes-Moscovici, i.e. only involving suitable residues of operator zeta functions. This problem is dealt with in the forthcoming paper [6], and we concentrate here on a local formula for the Hochschild class of $\mathrm{Ch}(\mathcal{A},\mathcal{M},D)$.

Since the normalized pairing between cyclic cohomology and $K-$theory is invariant under the operator S, it is natural to determine the dimension of the Chern character, i.e. the greatest $n \geq 1$ such that the analytic Chern-Connes character defined in the previous section is not in the range of the $S-$operation. Since we have $\mathrm{Im}(S) = \mathrm{Ker}(I)$, where $I : \mathrm{HC}^*(\mathcal{A}) \to \mathrm{HH}^*(\mathcal{A})$ is the natural forget map, it is important to have a computable local formula for the image $\mathrm{ICh}(\mathcal{A},\mathcal{M},D)$ of the Chern-Connes character in Hochschild cohomology. This would enable to prove or disprove that $\mathrm{Ch}(\mathcal{A},\mathcal{M},D)$ is or is not in the image of S. The local formula we get here is

$$< \mathrm{ICh}(\mathcal{A},\mathcal{M},D), a^0 \otimes \cdots \otimes a^p >= \int\!\!\!\!\!\!-_{\omega} \gamma a^0 [D,a^1] \cdots [D,a^p], \tag{17}$$

where ω is any state satisfying the assumptions (1) and (2) fixed in the first section. The RHS of this formula is local since it involves a Dixmier trace, and it is therefore more computable than the LHS.

We assume from now on, and in order to avoid unnecessary complications arising from the use of the Green operators, that $\mathrm{Ker}(D) = \{0\}$, but all the results remain true in the general case with the easy suitable modifications. Moreover, the local formula is stable under bounded perturbations of D. Recall that the analytic Chern-Connes character is a cyclic cohomology class ϕ over $\mathcal{A}$ which can be represented in the lowest dimension p by the cyclic cocycle:

$$\phi(a^0, ..., a^p) := \tau_s(\gamma a^0 [F,a^1] ... [F,a^p]),$$

where τ_s is the regularization (enables to win one degree of summability) defined by

$$\tau_s(\alpha) := \frac{1}{2}\tau(\gamma F[F,\alpha]),$$

and γ is the grading involution. The proof given in [17] extends easily to our setting and is omitted.

We fix from now on a state ω on $C_b(\mathbb{R})$ which vanishes on $C_0(\mathbb{R})$ and satisfies the assumptions fixed in the first section. For simplicity, we shall denote again for any $g \in C_b(\mathbb{R})$ by $\lim_{t\to\omega} g(t)$ the number $\omega(g)$ obtained out of g as a function of t and defined in Equation (1). Hence, in particular, this functional is invariant under dilations. We shall denote by τ_ω the Dixmier trace associated with the trace τ and the state ω, see Section 1 for more precise definitions.

Theorem 6. *The pairing of* $\mathrm{ICh}(\mathcal{M}, D)$ *with Hochschild homology coincides with the pairing of a local Hochschild cocycle* ϕ *given by the formula:*

$$< \phi, \sum_i a_0^i \otimes a_1^i ... \otimes a_p^i >:= \sum_i \int\!\!\!\!\!\!- \gamma a_0^i [D, a_1^i]...[D, a_p^i].$$

To be rigorous, this equality holds only up to constant since it depends on the normalizations of the Chern-Connes character and of the pairing (for a coherent choice of the normalizations, see [15]). The proof of Theorem 6 is based on some technical Lemmas that we state first.

Lemma 5. [16] *Let* $\omega = \sum_{i\in I} a_0^i \otimes ... \otimes a_p^i$ *be a Hochschild cycle. Let* f *be a compactly supported smooth even function such that* $f(0) = 1$*. Then*

$$< \mathrm{ICh}(\mathcal{A}, \mathcal{M}, D), \omega >= -\lim_{t\to 0} \sum_i \tau_s([f(tD), a_p^i] a_0^i [F, a_1^i]...[F, a_{p-1}^i]F).$$

Proof. Forget i but keep it in mind! Set $A = a_0[F, a_1]...[F, a_p]$. The function $f_t(x) = f(tx)$ converges simply to 1 when t goes to 0, and by the Lebesgue theorem, we deduce that $f(tD)$ converges weakly to the identity operator. Since τ is a normal trace and $\gamma F[F, A]$ is trace-class, we have:

$$< \mathrm{ICh}(\mathcal{A}, \mathcal{M}, D), \omega >= \tau(\gamma F[F, A]) = \lim_{t\to 0} \tau(f(tD)\gamma F[F, A]).$$

But, $f(tD)F = Ff(tD)$ and since f is an even function, we also have $f(tD)\gamma = \gamma f(tD)$. Hence we deduce:

$$< \mathrm{ICh}(\mathcal{A}, \mathcal{M}, D), \omega >= \lim_{t\to 0} \tau_s(f(tD)A).$$

Now the operator $f(tD)$ belongs to $L^1(\mathcal{M}, \tau)$ for any $t > 0$ and the operator a_p commutes with the grading involution γ, thus:

$$\tau_s(f(tD)A) = \tau_s(a_p f(tD)a_0[F, a_1]...[F, a_{p-1}]F) - \tau_s(f(tD)a_0[F, a_1]...[F, a_{p-1}]a_p F).$$

Set $\delta = [F, \cdot]$ for the derivation induced by F on $\mathcal{A}$ and let us apply the operator $id \otimes \delta \otimes ... \otimes \delta$ to the equality $b(\sum_i a_0^i \otimes a_1^i \otimes ... \otimes a_p^i) = 0$. We get:

$$\sum_i a_0^i [F, a_1^i]...[F, a_{p-1}^i] a_p^i = \sum_i a_p^i a_0^i [F, a_1^i]...[F, a_{p-1}^i],$$

which finishes the proof. □

Lemma 6. *Let* f *be an even function in* $C_c^\infty(\mathbb{R})$ *which equals 1 in a neighborhood of 0. Then* $\forall T \in \mathcal{M}, \forall a \in \mathcal{A}$ *we have:*

$$\lim_{\frac{1}{t}\to\omega} \tau([f(t^p D), a]T|D|^{-p+1}) = -p \int\!\!\!\!\!\!- [|D|, a]T.$$

Proof. Since f is even, this lemma only involves $|D|$ and we can assume $D \geq 0$. Let us first formally replace $[f(tD), a]$ by $f'(tD)[tD, a]$. We obtain:

$$\lim_{t^{-p}\to\omega} \tau([f(tD), a]TD^{-p+1}) = \lim_{t^{-p}\to\omega} \tau(tD^{-p+1} f'(tD)[D, a]T)$$

But setting $f_1(x) = x^{-p+1}f'(x)$ if $x \geq 0$ and extending f_1 to an even function, we obtain a well defined function f_1 satisfying the assumptions of Lemma 5. So:

$$\lim_{t^{-p}\to\omega} \tau([f(tD), a]TD^{-p+1}) = \lim_{t^{-p}\to\omega} t^p \tau(g(t|D|)[|D|, a]T).$$

and the proof is complete. It remains thus to show that:

$$A(t) = [f(tD), a] - f'(tD)[tD, a]$$

belongs to $L^{p,\infty}(\mathcal{M}, \tau)$ and that $\|A(t)\|_{p,\infty}$ is an $O(t)$. We now use the Fourier transform:

$$[f(tD), a] = \int_{\mathbb{R}} \hat{f}(u)[e^{iutD}, a]du.$$

But recall that $[e^{isD}, a] = \int_0^1 e^{isvD} is[D, a]e^{is(1-v)D}dv$, and so:

$$f'(tD)[tD, a] = \int_{\mathbb{R}} iu\hat{f}(u)e^{iutD}[tD, a]du.$$

Thus

$$\begin{aligned} A(t) = \int_{\mathbb{R}} \hat{f}(u) \int_0^1 & e^{iutsD}[iutD, a]e^{iut(1-s)D} - e^{iutD}[iutD, a])dsdu \\ &= \int_{\mathbb{R}} \hat{f}(u) \int_0^1 e^{iutsD}[[iutD, a], e^{iut(1-s)D}]dsdu \\ &= t^2 \int_{\mathbb{R}} u^2 \hat{f}(u) \int_0^1 \int_0^1 (1-s)e^{iut(s+(1-s)r)D}[D, [D, a]]e^{iut(1-s)(1-r)D}dsdrdu. \end{aligned}$$

Note that $[D, [D, a]]$ is bounded since we could assumed $D = |D|$ in this proof. Thus $\|A(t)\|_{\mathcal{M}} = O(t^2)$.

On the other hand, we have:

$$|\tau(A(t)TD^{-p+1})| \leq \|T\| \int_0^{+\infty} \mu_s(A(t))\mu_s(D^{-p+1})ds \leq C\|T\| \int_0^{+\infty} \mu_s(A(t)) \frac{ds}{(1+s)^{(p-1)/p}},$$

where C is some constant. Let us see that there exists a constant $K > 0$ such that

$$\mu_s(A(t)) = 0, \quad \forall s > K/t^p.$$

To this end, let $A > 0$ be such that $\mathrm{Supp}(f) \subset [-A, A]$ and denote by E_t the spectral projection of D^{-1} corresponding to the interval $[t/A, +\infty)$. We have

$$\tau(E_t) = |\{s > 0, \mu_s(D^{-1}) \geq t/A\}|.$$

Since there exists a constant $B > 0$ such that $\mu_s(D^{-1}) \leq B/s^{1/p}$, we get

$$\tau(E_t) \leq |\{s > 0, s^{1/p} \leq AB/t\}|,$$

and hence $\tau(E_t) = O(t^{-1/p})$.

But,

$$\begin{aligned} \mu_s(A(t)) &\leq \mu_{s/2}(f(tD)a - f'(tD)[tD, a]) + \mu_{s/2}(af(tD)) \\ &= \mu_{s/2}(E_t(f(tD)a - f'(tD)[tD, a])) + \mu_{s/2}(af(tD)E_t) \\ &\leq \|f(tD)a - f'(tD)[tD, a]\|\mu_{s/2}(E_t) + \|af(tD)\|\mu_{s/2}(E_t). \end{aligned}$$

The first and the last inequalities follow from [26][Lemma 2.5, (v) and (vi), page 276]. Since $\tau(E_t) = O(t^{-1/p})$, there exists a constant $K > 0$ such that

$$\mu_{s/2}(E_t) = 0 \text{ for } s > K/t^p,$$

and hence

$$\mu_s(A(t)) = 0 \text{ for } s > K/t^p.$$

It follows that

$$|\tau(A(t)TD^{-p+1})| \leq C\|T\|t^2 \int_0^{K/t^p} \frac{ds}{(1+s)^{(p-1)/p}},$$

and hence $\tau(A(t)TD^{-p+1}) \to 0$ when $t \to 0$. $\square$

Proposition 7. *The pairing* $\mathrm{ICh}(\mathcal{A}, \mathcal{M}, D)$ *with Hochschild homology is given by*

$$< \mathrm{ICh}(\mathcal{A}, \mathcal{M}, D), \sum_i a_0^i \otimes a_1^i \cdots \otimes a_p^i >= -p \sum_i ⨍ \gamma[|D|, a_p^i]a_0^i[F, a_1^i]...[F, a_{p-1}^i]D^{p-1}.$$

Proof. Let f be an even function in $C_c^\infty(\mathbb{R})$ which equals 1 in a neighborhood of 0 as above. Then Proposition 5 shows that

$$< \mathrm{ICh}(\mathcal{A}, \mathcal{M}, D), \sum_i a_0^i \otimes a_1^i \cdots \otimes a_p^i >= -\sum_i \lim_{t\to 0} \tau_s([f(tD), a_p^i]a_0^i[F, a_1^i]...[F, a_{p-1}^i]F).$$

Lemma 6 then gives with the bounded operator $T = a_0[F, a_1]...[F, a_{p-1}]F\gamma|D|^{p-1}$:

$$\begin{aligned} -\lim_{t\to 0} \tau_s([f(tD), a_p^i]a_0^i[F, a_1^i] \cdots [F, a_{p-1}^i]F) &= -\lim_{t\to 0} \tau([f(tD), a_p^i]a_0^i[F, a_1^i] \cdots [F, a_{p-1}^i]F\gamma|D|^{p-1}|D|^{-p+1}) \\ = p ⨍ [|D|, a_p^i]a_0^i[F, a_1^i] \cdots [F, a_{p-1}^i]F\gamma|D|^{p-1} &= -p ⨍ [|D|, a_p^i]a_0^i[F, a_1^i] \cdots [F, a_{p-1}^i]\gamma D^{p-1} \\ &= -p ⨍ \gamma[|D|, a_p^i]a_0^i[F, a_1^i] \cdots [F, a_{p-1}^i]D^{p-1}. \end{aligned}$$

This completes the proof. $\square$

Let δ be the derivation on the algebra $\tilde{\mathcal{A}}$ generated by $\mathcal{A}$ and $[|D|, \mathcal{A}]$ given by $\delta(X) = [|D|, X]$. Recall that for any Hochschild cocycle ρ on $\mathcal{B}$, one defines a new Hochschild cocycle by setting [32]:

$$< \delta_j(\rho), (X_0, X_1, \cdots, X_p) >= \hat{\rho}(X_0 dX_1 \cdots dX_{j-1}\delta(X_j)dX_{j+1} \cdots dX_p),$$

where $\hat{\rho}$ is the corresponding non commutative current [17]. Moreover, it is well known and easy to check that $\delta_j(\rho) + \delta_{j+1}(\rho)$ is a Hochschild coboundary. Therefore, the derivation δ induces, by considering the class of $(-1)^j\delta_j(\rho)$ for any $j \in \{1, \cdots, p\}$, a map between Hochschild cohomology:

$$i_\delta : \mathrm{HH}^*(\tilde{\mathcal{A}}) \longrightarrow \mathrm{HH}^{*+1}(\tilde{\mathcal{A}}).$$

Moreover, this map satisfies the relation $i_\delta^2 = 0$ in Hochschild cohomology.

Proposition 8.

(1) *Recall that* p *is even. We define Hochschild cocycles on* $\mathcal{A}$ *by setting*

$$\Phi(a^0, \cdots, a^p) = \tau_\omega(\gamma a^0[D, a^1] \cdots [D, a^p]D^{-p}), \Psi(a^0, \cdots, a^p) = p\tau_\omega(\gamma a^0[F, a^1] \cdots [F, a^{p-1}][|D|, a^p]D^{-1})$$

$$\text{and } \varphi_k(a^0, \cdots, a^{p-1}) = \tau_\omega(\gamma a^0[F, a^1] \cdots [F, a^k]D^{p-k}), k = 0 \cdots p.$$

Moreover, $\varphi_p = 0$.

(2) *The Hochschild cocycles* Φ *and* Ψ *are cohomologous in Hochschild cohomology.*

Proof.

(1) We have by straightforward computation:

$$b\Phi(a^0, \cdots, a^{p+1}) = (-1)^p \left[\tau_\omega(\gamma a^0[D, a^1] \cdots [D, a^p][a^{p+1}, D^{-p}])\right].$$

But $[a^{p+1}, D^{-p}]$ belongs to the ideal $L^1(\mathcal{M}, \tau)$, since we have:

$$[a^{p+1}, |D|^{-1}] = -|D|^{-1}[a^{p+1}, |D|]|D|^{-1},$$

and $[a^{p+1}, D^{-p}] = \sum_j |D|^{-j}[a^{p+1}, |D|^{-1}]|D|^{-(p-j-1)}$.

In the same way, we have:

$$b\Psi(a^0, \cdots, a^{p+1}) = -p(-1)^p\tau_\omega(\gamma a^0[F, a^1] \cdots [F, a^{p-1}][|D|, a^p][a^{p+1}, D^{-1}]).$$

But again, $[a^{p+1}, D^{-1}]$ belongs to $L^p(\mathcal{M}, \tau)$ and thus the operator $\gamma a^0[F,a^1]\cdots[F,a^{p-1}][|D|,a^p][a^{p+1},D^{-1}]$ is trace class. Therefore $b\Psi = 0$. Thus, $b\Phi = 0$. The proof for the cochains φ_k is similar. Now we obviously have:

$$\tau_\omega(\gamma a^0[F,a^1]\cdots[F,a^p]) = \tau_\omega(\gamma F[F,a^0][F,a^1]\cdots[F,a^p]) = 0.$$

(2) We first point out that

$$[|D|^{-1},[D,a]] = -|D|^{-1}[|D|,[D,a]]|D|^{-1}.$$

Therefore, in the expression of $\Phi(a^0,\cdots,a^p)$ we can move $|D|^{-1}$ to the left and in particular:

$$\Phi(a^0,\cdots,a^p) = \tau_\omega(\gamma a^0[D,a^1]|D|^{-1}\cdots[D,a^p]|D|^{-1}). \tag{18}$$

On the other hand,

$$[D,a]|D|^{-1} = [F,a] + F[|D|,a]|D|^{-1} = [F,a] + [|D|,a]D^{-1} + [F,[|D|,a]]|D|^{-1}.$$

Now, since $F = D|D|^{-1}$, we deduce

$$[F,[|D|,a]] = [D,[|D|,a]]|D|^{-1} - F[|D|,[|D|,a]]|D|^{-1}.$$

Therefore and since $[D,[|D|,a]] = [|D|,[D,a]]$ is bounded, the operator $[F,[|D|,a]]$ belongs to $L^{p,\infty}(\mathcal{M},\tau)$. Therefore,

$$[D,a]|D|^{-1} - ([F,a] + [|D|,a]D^{-1}) \quad \in L^p(\mathcal{M},\tau).$$

Hence we can replace $[D,a]|D|^{-1}$ by $[F,a]+[|D|,a]D^{-1}$ when necessary in the expression of Φ in Equation (18). On the other hand, we have:

$$\delta_j(\varphi)(a^0,\cdots,a^p) = \tau_\omega(\gamma a^0[F,a^1]\cdots[F,a^{j-1}][|D|,a^j][F,a^{j+1}]\cdots[F,a^p]D^{-1}).$$

Thus and since $(-1)^j\delta_j(\varphi)$ is cohomologous to $\delta_p(\varphi) = \Phi$, we deduce that:

$$\Phi(a^0,\cdots,a^p) = \sum_{j=1}^p \tau_\omega(\gamma a^0[F,a^1]\cdots[F,a^{j-1}][|D|,a^j]D^{-1}[F,a^{j+1}]\cdots[F,a^p]) + R_p(a^0,\cdots,a^p),$$

where $R_p(a^0,\cdots,a^p)$ corresponds to the terms where the factor $[|D|,a]D^{-1}$ appears at least twice. The first remark is that

$$D^{-1}[F,a] + [F,a]D^{-1} = |D|^{-1}\delta^2(a)|D|^{-2} - D^{-1}\delta([D,a])|D|^{-2}. \tag{19}$$

Therefore, we have:

$$\tau_\omega(\gamma a^0[F,a^1]\cdots[F,a^{j-1}][|D|,a^j]D^{-1}[F,a^{j+1}]\cdots[F,a^p]) = \\ (-1)^j\tau_\omega(\gamma a^0[F,a^1]\cdots[F,a^{j-1}][|D|,a^j][F,a^{j+1}]\cdots[F,a^p]D^{-1}).$$

This latter is nothing but a representative for $i_\delta(\varphi)(a^0,\cdots,a^p)$ for any j. Therefore:

$$\Phi(a^0,\cdots,a^p) = p\tau_\omega(\gamma a^0[F,a^1]\cdots[F,a^{p-1}][|D|,a^p]D^{-1}) + R_p(a^0,\cdots,a^p) + b\alpha(a^0,\cdots,a^p),$$

for some cochain α. To finish the proof, we thus need to show that R_p is a coboundary. But this is a consequence of the fact that $i_\delta^2 = 0$ in Hochschild cohomology. More precisely, consider for instance the Hochschild cocycle

$$\varphi_2(a^0,\cdots,a^{p-2}) = \tau_\omega(\gamma a^0[F,a^1]\cdots[F,a^{p-2}]D^{-2}).$$

Then $i_\delta^2\varphi$ can be represented for $1 \le j < k \le p$ by the Hochschild cocycle

$$(a^0,\cdots,a^p) \longmapsto \\ (-1)^{jk}\tau_\omega(\gamma a^0[F,a^1]\cdots[F,a^{j-1}][|D|,a^j][F,a^{j+1}]\cdots[F,a^{k-1}][|D|,a^k][F,a^{k+1}]\cdots[F,a^p]D^{-2}).$$

But again, this is precisely,

$$\tau_\omega(\gamma a^0[F,a^1]\cdots[F,a^{j-1}][|D|,a^j]D^{-1}[F,a^{j+1}]\cdots[F,a^{k-1}][|D|,a^k]D^{-1}[F,a^{k+1}]\cdots[F,a^p]),$$

and hence corresponds to the $(j,k)-$term in the expression of R_p. Thus all the terms where $[|D|,a]D^{-1}$ appears twice are coboundaries. The same argument using the Hochschild cocycles $(\varphi_k)_{k\geq 3}$ shows that all the other terms in R_p are coboundaries. The proof is thus complete.

□

Proof of Theorem 6.
By using Proposition 7 and Proposition 8, it is sufficient to show that

$$\tau_\omega(\gamma a^0[F,a^1]\cdots[F,a^{p-1}][|D|,a^p]D^{-1}) = -\tau_\omega(\gamma[|D|,a^p]a^0[F,a^1]\cdots[F,a^{p-1}]D^{-1}).$$

But

$$\gamma[|D|,a^p]+[|D|,a^p]\gamma = 0,$$

and hence

$$\tau_\omega(\gamma[|D|,a^p]a^0[F,a^1]\cdots[F,a^{p-1}]D^{-1}) = -\tau_\omega([|D|,a^p]\gamma a^0[F,a^1]\cdots[F,a^{p-1}]D^{-1}).$$

Therefore and using the trace property of τ_ω, we get:

$$\tau_\omega(\gamma[|D|,a^p]a^0[F,a^1]\cdots[F,a^{p-1}]D^{-1}) = -\tau_\omega(\gamma a^0[F,a^1]\cdots[F,a^{p-1}]D^{-1}[|D|,a^p]).$$

Again we can use Equation (19) to move D^{-1} to the right and this completes the proof.

Remark 6. The equivariant case with respect to an action of a compact Lie group can be handled in the same way using for instance the definitions of [3]. The local formula obtained for the equivariant Hochschild chern character is then interesting in view of fixed point theorems because it gives in the usual situations such as compact manifolds and compact foliated manifolds a measure on the cosphere or longitudinal cosphere bundle which is supported by the fixed points.

Remark 7. In many interesting situations (e.g.: measured foliations, coverings, almost periodic operators), the local expression of the Hochschild class of the Chern-Connes character obtained in Theorem 6 actually defines a cyclic cocycle on $\mathcal{A}$.

If we define the von Neumann Yang-Mills action $\mathrm{YM}^\tau(\nabla)$ associated with any compatible connection ∇ on a finitely generated projective module Ω over the $*$-algebra $\mathcal{A}$, then we can state following Connes' method (See [15], Theorem 4, page 561):

Proposition 9. *Let $(\mathcal{A},\mathcal{M},D)$ be a four dimensional spectral triple. Set*

$$\phi^\tau_\omega(a^0,a^1,a^2,a^3,a^4) = 2\tau_\omega(\gamma a^0[D,a^1][D,a^2][D,a^3][D,a^4]D^{-4})$$

and assume that ϕ^τ_ω is cyclic. Then for any hermitian finitely generated projective module Ω over $\mathcal{A}$ with a compatible connection ∇, we have

$$|<[\Omega],\phi^\tau_\omega>| \leq \mathrm{YM}^\tau(\nabla).$$

The consequences of this proposition in the examples listed before will be treated in a forthcoming paper. We point out that the case of almost periodic operators is especially interesting for applications to quasiperiodic tillings. In the example of measured foliated manifolds, the representative of the Hochschild class of the Chern-Connes character given above, is a cyclic cocycle over $C^\infty(M)$. The computations in [6] give the following expected result:

Theorem 7. *The current C in M which represents the Hochschild class of the Chern-Connes character of the Dirac operator along the leaves via the Connes-Hochschild-Kostant-Rosenberg isomorphism is, up to constant, the Ruelle-Sullivan current associated with the holonomy invariant transverse measure Λ.*

To end this section, let us state the positivity result obtained in the case of foliations by four dimensional spin manifolds. Let D be the Dirac operator along the leaves and let γ be the $\mathbb{Z}_2$-grading. The following proposition is a corollary of Proposition 9 and can be understood as furnishing a new inequality on the manifold M for any given measured foliation.

Proposition 10. *For any hermitian vector bundle E over M and any hermitian connection ∇ on E, we have*

$$| < c_1(E)^2/2 - c_2(E), [C_\Lambda] > | \leq \mathrm{YM}^\Lambda(\nabla)$$

where $c_1(E)$ and $c_2(E)$ are the Chern classes of E.

Proof. If we compute the Chern character of E and integrate it against the Ruelle-Sullivan current then we obtain the pairing of our Hochschild cocycle (which is cyclic here) with the K-theory of M by Theorem 7. Now Proposition 9 enables to conclude. □

Appendix A. Singular numbers

We gather in this appendix some general facts about Dixmier traces associated with type II von Neumann algebras. We shall denote by $\mathcal{M}$ a von Neumann algebra acting on a Hilbert space H, and we shall assume that there exists a positive normal semi-finite faithful trace τ on $\mathcal{M}$.

A.1. **τ-measurable operators.** A densely defined closed operator T acting on H is said to be τ-measurable if it is affiliated with $\mathcal{M}$ and if there exists, for each $\epsilon > 0$, a projection E in $\mathcal{M}$ such that $E(H) \subset Dom(T)$ and $\tau(1-E) \leq \epsilon$. Let $T = U|T|$ be the polar decomposition of the densely defined closed operator T, and denote by

$$|T| = \int_0^{+\infty} \lambda dE_\lambda$$

the spectral decomposition of its module. Then, the operator T is τ-measurable if and only if both U and the $E'_\lambda s$ $(\lambda \in \mathbb{R}_+^*)$ belong to $\mathcal{M}$, and $\tau(1-E_\lambda) < +\infty$ for λ large enough. Let us also recall that the set of all τ-measurable operators is a $*$-algebra with respect to the strong sum, the strong product, and the adjoint of (densely defined) closed operators (cf. [12]).

A.2. **τ-singular numbers.** For any $t > 0$, the t^{th} singular number (s-number) $\mu_t(T)$ of a τ-measurable operator T is defined by :

$$\mu_t(T) = \mathrm{Inf}\{\|TE\|, E = E^2 = E^* \in \mathcal{M} \text{ and } \tau(1-E) \leq t\}.$$

Thanks to the τ-measurability of T, we have :

$$0 \leq \mu_t(T) = \mu_t(|T|) < +\infty$$

for any $t > 0$. There are several equivalent definitions of the singular numbers. For instance, we have (cf [26]):

$$\mu_t(T) = \mathrm{Inf}\{\lambda \geq 0, \tau(1-E_\lambda) \leq t\},$$

where $|T| = \int_0^{+\infty} \lambda dE_\lambda$ still denotes the spectral decomposition of $|T|$, a fact which shows that the function $t \to \mu_t(T)$ is nothing but the non-increasing rearrangement of $|T|$ as a positive measurable function on the measure space $(sp(|T|) \setminus \{0\}, m)$. Here, $sp(|T|)$ denotes the spectrum of $|T|$ and m the spectral measure defined by

$$m(B) = \tau(1_B(|T|)), (B \text{ Borel subset of } sp(|T|) \setminus \{0\}).$$

Note also that we have for any $t > 0$:

$$\mu_t(T) = dist(T, \mathcal{R}_t),$$

where $\mathcal{R}_t$ is the set of all τ-measurable operators S such that $\tau(supp(|S|)) \leq t$. This equality shows that the s-numbers may be viewed as a natural extension of the classical approximation numbers.

For a detailed study of the generalized s-numbers, we refer to [26], where several spectral inequalities are proved. Let us just mention here the most useful of them, for the convenience of the reader:

Lemma 7. *(i) For any τ-measurable operator T, the function $t \to \mu_t(T)$ is non increasing and right continuous. Moreover, $\mu_t(T) \to \|T\|$ when $t \to 0$;*

(ii) For any τ-measurable operator T, any complex number λ, and any $t > 0$, we have:

$$\mu_t(T) = \mu_t(|T|) = \mu_t(T^*) \text{ and } \mu_t(\lambda T) = |\lambda| \mu_t(T)$$

(iii) For any τ-measurable operator T and non decreasing right-continuous function f on $[0,+\infty)$ such that $f(0) \geq 0$, we have:

$$\mu_t(f(|T|)) = f(\mu_t(T)), \quad \forall t > 0;$$

(iv) For any pair of τ-measurable operators T, S and for any $t, s > 0$, we have:

$$\mu_{t+s}(T+S) \leq \mu_t(T) + \mu_s(S) \text{ and } \mu_{t+s}(TS) \leq \mu_t(T)\mu_s(S);$$

(v) For any τ-measurable operator T, any pair of operators $A, B \in \mathcal{M}$ and any $t > 0$, we have :

$$\mu_t(ATB) \leq \|A\|\mu_t(T)\|B\|;$$

(vi) For any pair of τ-measurable operators T, S satifying $T \leq S$, we have:

$$\forall t > 0, \mu_t(T) \leq \mu_t(S).$$

A.3. **Non commutative integration theory.** Let T be a τ-measurable operator. For any continuous increasing function f on $[0,+\infty)$ with $f(0) = 0$, we have:

$$\tau(f(|T|)) = \int_0^\infty f(\mu_t(T))dt.$$

[26][Corollary 2.8, page 278]. This basic relation explains why the s-numbers are of interest in the study of non commutative Banach spaces of functions such as $L^p(\mathcal{M}, \tau)$, $p \geq 1$. In particular, we have

$$T \in L^p(\mathcal{M}, \tau) \Leftrightarrow [t \to \mu_t(T)] \in L^p([0, \infty))$$

and

$$\|T\|_p = (\int_0^\infty \mu_t(T)^p dt)^{1/p}.$$

Most of the known s-numbers inequalities are based on the properties of the following function

$$\sigma_t(T) = \int_0^t \mu_s(T)ds, \quad s > 0.$$

The following lemma gives three equivalent expressions of $\sigma_t(T)$.

Lemma 8. *Let T be a τ-measurable operator. For any $t > 0$ we have*

$$\sigma_t(T) = inf\{\|T_1\|_1 + t\|T_2\|_\infty, T = T_1 + T_2, T_1 \in L^1(\mathcal{M}, \tau), T_2 \in \mathcal{M}\};$$

and if $\mathcal{M}$ has no minimal projections, then we have:

$$\sigma_t(T) = sup\{\tau(E|T|E), E \in \mathcal{M}, E^2 = E^* = E, \tau(E) \leq t\}.$$

Proof. The first interpolation formula is proved in [26], page 289 and the third equality also goes back to [26]. ☐

Proposition 11.

If T_1, T_2 are two positive τ-measurable operators then for $(t_1, t_2) \in \mathbb{R}_+^*$, we have

$$\sigma_{t_1}(T_1) + \sigma_{t_2}(T_2) \leq \sigma_{t_1+t_2}(T_1 + T_2) \text{ and } \sigma_t(T_1 + T_2) \leq \sigma_t(T_1) + \sigma_t(T_2).$$

Proof. By imbeding $\mathcal{M}$ in $\tilde{\mathcal{M}} = \mathcal{M} \otimes L^\infty([0,1], dt)$ and using the simple fact that $\mu_t^{\mathcal{M}}(T) = \mu_t^{\tilde{\mathcal{M}}}(T \otimes id)$ we can assume that $\mathcal{M}$ has no minimal projections. If E_1, E_2 are two projections in $\mathcal{M}$ such that $\tau(E_1) = t_1$ and $\tau(E_2) = t_2$ then the projection $E = E_1 \vee E_2$ belongs to $\mathcal{M}$ and satisfies $\tau(E) \leq t_1 + t_2$. We thus have

$$\tau((T_1 + T_2)E) = \tau(T_1 E) + \tau(T_2 E) \geq \tau(T_1 E_1) + \tau(T_2 E_2),$$

thus lemma 8 gives the first inequality.

The second one follows similarily from lemma 7 (iii), see for instance [26][Theorem 4.4 (ii)]. ☐

References

[1] M. F. Atiyah, *Elliptic operators, discrete groups and von Neumann algebras.*, Astérisque 32/33, SMF, 1976 (1989), 79–92.
[2] M.F. Atiyah and I. M Singer, *The index of elliptic operators, I*, Anna. Math. **87** (1968), 484–530.
[3] M. Benameur, *Foliated group actions and cyclic cohomology*, preprint.
[4] M. Benameur and V. Nistor, *Homology of complete symbols and non Commutative geometry*, "Collected papers on Quantization of Singular Symplectic Quotients", Progress in Math. 198.
[5] M. Benameur and V. Nistor, *Homology of algebras of families of pseudodifferential operators*, to appear in J. of Funct. Analysis.
[6] M. Benameur, T. Fack and V. Nistor, *The local measured index theorem for foliations*, work in progress.
[7] N. Berline, E. Getzler and M. Vergne, *Heat kernels and Dirac operators*, Springer-Verlag 1991.
[8] Bourbaki, *Integration* Fasc. 25, livre 6, chapitre 6.
[9] M. Breuer, *Fredholm theories in von Neumann algebras. I*, Math. Ann. 178 (1968), 243-254.
[10] M. Breuer, *Fredholm theories in von Neumann algebras. II*, Math. Ann. 180 (1969), 313-325.
[11] A. Carey and J. Phillips, *Unbounded Fredholm modules and spectral flow*, Can. J. Math. Vol. 50, (1998), 673-718.
[12] L.A. Coburn, R.D. Moyer and I.M. Singer, *C^*-algebras of almost periodic pseudo-differential operators*, Acta Math. 130 (1973), 279-307.
[13] A. Connes. *Sur la théorie non commutative de l'integration.* Algèbres d'opérateurs, LNM 725, Springer Verlag, 19–143, 1979.
[14] A. Connes. *The action functional in noncommutative geometry.* Comm. Math. Phys. 117, (1988), 673–683.
[15] A. Connes. *Noncommutative Geometry.* Academic Press, New York - London, 1994.
[16] A. Connes. *Cours au Collège de France*, 1990.
[17] A. Connes, *Non-commutative differential geometry*, Publ. IHES 62, 257-360 (1986).
[18] A. Connes and J. Cuntz, *Quasi homomorphismes, cohomologie cyclique et positivité*, Commun. Math. Phys. 114, 515-526 (1988).
[19] A. Connes and H. Moscovici, *The local index formula in nnoncommutative geometry*, Geom Funct. An. Vol 5, No 2 (1995).
[20] A. Connes and H. Moscovici, *Hopf algebras, cyclic cohomology and the transverse index theorem*, preprint.
[21] J. Cuntz, *Bivariante K-Theorie fr lokalkonvexe Algebren und der Chern-Connes-Charakter. (German)* [Bivariant K-theory for locally convex algebras and the Chern-Connes character], Doc. Math. 2 (1997), 139–182.
[22] J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien (Algèbres de von Neumann)*, Gauthier-Villars, Paris, 1957.
[23] J. Dixmier, *Existence de traces non normales*, CRAS Paris 262 (1966), A1107-A1108.
[24] K. J. Dykema, N. J. Kalton, *Spectral characterization of sums of commutators II.*, J. Reine Angew. Math. 504 (1998), 127–137.
[25] T. Fack, *Sur la notion de valeur caractéristique.* JOT 7, (1982), 307-333.
[26] T. Fack and H. Kosaki, *Generalized s-numbers of measured operators*, Pac. J. Math. 123 (1986), 269-300.
[27] D. Guido and T. Isola, *Singular traces on semifinite von Neumann algebras*, J. funct. Anal. 134 (1995), 451-485.
[28] D. Guido and T. Isola, *Noncommutative Riemann integration and Novikov-Shubin invariants for open manifolds*, J. funct. Anal. 176 (2000), 115-152.
[29] R. Haag, *Local quantum physics* Fields, particles, algebras. Second edition. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1996.
[30] C. Kassel, *Le résidu non commutatif*, Séminaire Bourbaki, 1988-89, No 708.
[31] J-L. Loday, *Cyclic homology*, Springer-Verlag, Berlin-Heidelberg-New York, 1992.
[32] MacLane, *Homology*, Springer-Verlag, Berlin-Heidelberg-New York, 1995.
[33] J. Milnor, *Introduction to algebraic K-theory*, Ann. of Math. Stud., 72, Princeton Univ. Press (1971).
[34] C.C Moore and C. Schochet, *Global analysis on Foliated spaces*, Springer-Verlag 1988.
[35] V. Nistor and A. Weinstein, and Ping Xu, *Pseudodifferential operators on groupoids*, Penn State Preprint No. 202 (1997).
[36] J-F. Plante, *Foliations with measure preserving holonomy*, Ann. Math. 102, 327-361 (1975).
[37] R. Prinzis, *Traces résiduelles et asymptotique du spectre d'opérateurs pseudo-différentiels.*, Thèse de Doctorat de l'Université de Lyon, 1995.
[38] D. Ruelle and D. Sullivan *Currents, flows and diffeomorphisms*, Topology 14, 319-327 (1975).
[39] M. A. Shubin,*Pseudodifferential almost periodic operators and von Neumann algebras*, Trnans. Moscow Math. Soc. 1 (1979) 103-166.
[40] M. A. Shubin,*The spectral theory and the index of elliptic operators with almost periodic operators*, Russian Math. Survey 34:2 (1979) 109-157.
[41] B. Tsygan,*Homology of matrix Lie algebras over rings and Hochschild homology*, Uspekhi Math. Nauk., **38** (1983), 217-218.
[42] J. M. Gracia-Bondia, J. C. Varilly and H. Figueroa, *Elements of Noncommutative Geometry*, Birkhauser advanced texts, (2001).
[43] D. V. Widder, *The Laplace transform*, Princeton Univ. Press (1946).

Inst. Desargues, Lyon, France
E-mail address: benameur@igd.univ-lyon1.fr

Inst. Desargues, Lyon, France
E-mail address: fack@igd.univ-lyon1.fr